NORM ATTAINING OPERATORS WHICH SATISFY A BOLLOBÁS TYPE THEOREM

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Abstract. In this paper we are interested to study the set $\mathcal{A}_{\|\cdot\|}(X,Y)$ of all norm one linear operators T from X into Y which attain the norm and satisfy the following: given $\varepsilon > 0$, there exists η, which depends on ε and T, such that if $\|T(x)\| > 1-\eta$, then there is x_0 such that $\|x_0 - x\| < \varepsilon$ and T itself attains the norm at x_0. We show that every norm one functional on c_0 which attains the norm belongs to $\mathcal{A}_{\|\cdot\|}(c_0,\mathbb{K})$. Also, we prove that the analogous result is not true neither for $\mathcal{A}_{\|\cdot\|}(\ell_1,\mathbb{K})$ nor $\mathcal{A}_{\|\cdot\|}(\ell_\infty,\mathbb{K})$. Under some assumptions, we show that the sphere of the compact operators belongs to $\mathcal{A}_{\|\cdot\|}(X,Y)$ and that this is no longer true when some of these hypotheses are dropped. We show also that the natural projections on c_0 or ℓ_p, for $1 \leqslant p < \infty$, always belong to this set. The analogous set $\mathcal{A}_{\mathrm{nu}}(X)$ for numerical radius of an operator instead of its norm is also defined and studied. We give non trivial examples of operators on infinite dimensional Banach spaces which belong to $\mathcal{A}_{\|\cdot\|}(X,X)$ but not to $\mathcal{A}_{\mathrm{nu}}(X)$ and vice-versa. Finally, we establish some techniques which allow to connect both sets.

1. Introduction

Over the past few years, some Bishop-Phelps-Bollobás type properties were defined and studied. Some of them are stronger than the Bishop-Phelps-Bollobás property (**BPBp**, for short), introduced eleven years ago by María Acosta, Richard Aron, Domingo García, and Manuel Maestre (see [3]) in order to study the operator version of the Bollobás' theorem, which, roughly speaking, means that if $\langle x^*, x\rangle \approx 1$, then there exist new elements y^* and y such that $y^* \approx x^*$, $y \approx x$, and y^* attains the norm at y. One of these properties is the so-called Bishop-Phelps-Bollobás *operator* property (**BPBop**, for short), studied in [7, 8, 11]: a pair of Banach spaces (X,Y) satisfies the **BPBop** if given $\varepsilon > 0$, then there is $\eta = \eta(\varepsilon) > 0$ such that whenever an operator T with $\|T\| = 1$ from X into Y and a point x satisfy $\|T(x)\| > 1-\eta$, then there exists a new element x_0 such that $\|x_0 - x\| < \varepsilon$ and T itself attains its norm at x_0. Notice that it seems to be very restrictive at first glance since, in this case, we do not change the operator which almost attains the norm. Indeed, after some negative results presented in [7] and [11], it was finally proved in [8] that if $\dim(X), \dim(Y) \geqslant 2$, then the pair (X,Y) can not satisfy this property. On the other hand, [11, Theorem 2.1] gives a characterization for it in terms of linear functionals. A Banach space X is uniformly convex if and only if the pair $(X,\mathbb{K})$ satisfies the **BPBop**.

Let us observe that the **BPBop** and **BPBp** can be viewed as uniform properties in the sense that the $\eta > 0$ that appears in their definitions depends just on a given positive number $\varepsilon > 0$. Since there is no hope for the vector valued case, it was studied a local version of it in [7, 15, 16], when the η depends on a given $\varepsilon > 0$ and also on a fixed norm one operator T, that is, given $\varepsilon > 0$ and a norm one operator T, there is $\eta = \eta(\varepsilon, T) > 0$ such that if x is such that $\|T(x)\| > 1-\eta$, then there is x_0 with $\|x_0 - x\| < \varepsilon$ and T attains the norm at x_0. By the results in [7, 15, 16], it

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brings a lot of differences between the local and uniform properties. Now, we have many positive examples about it, not just in the functional case, but also in the operator case (see also [9]).

In this paper, we investigate the set $\mathcal{A}$ of all norm one operators T which attain the norm and enjoy the property that for a given $\varepsilon > 0$, there is $\eta = \eta(\varepsilon, T) > 0$ such that if $\|T(x)\| > 1 - \eta$ for some $x \in S_X$, then there is $x_0 \in S_X$ with $\|x_0 - x\| < \varepsilon$ and T attains the norm at x_0. Let us notice the obvious fact that if a pair of Banach spaces (X, Y) satisfies the (local) **BPBop**, then every norm one operator T from X into Y which attains the norm belongs to the set $\mathcal{A}$. It is worth mentioning also that if a pair (X, Y) satisfies the (local) **BPBop**, then the domain space X must be reflexive by James' theorem. However, we present concrete examples of non trivial operators T which satisfies the above property even when X is a non reflexive space. Moreover, we study the equivalent set by considering the numerical radius of T instead of its norm. We will give the precise definition of these sets later. Now, let us restart the introduction with all the necessary definitions and background we need.

Let X, Y be Banach spaces over a scalar field $\mathbb{K}$, which can be the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We denote by B_X and S_X the closed unit ball and the unit sphere of a Banach space X, respectively. The topological dual space of X is denoted by X^* and the action of an element $x^* \in X^*$ at an element $x \in X$ is denoted by $\langle x^*, x \rangle$. As usual, the set $\mathcal{L}(X, Y)$ stands for the bounded linear operators from X into Y and we simply denoted it as $\mathcal{L}(X)$ when $Y = X$. We say that T attains the norm or T is a norm attaining operator when there is $x_0 \in B_X$ such that $\|T(x_0)\| = \|T\|$. To deal with the numerical radius of an operator, we introduce the set $\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : \langle x^*, x \rangle = 1\}$. Given $T \in \mathcal{L}(X)$, the numerical radius of T is defined as $\nu(T) := \sup\{|\langle x^*, T(x) \rangle| : (x, x^*) \in \Pi(X)\}$. We say that T attains the numerical radius or T is a numerical radius attaining operator when there is $(x_0, x_0^*) \in \Pi(X)$ such that $|\langle x_0^*, T(x_0) \rangle| = \nu(T)$. For a background on this topic, we refer the reader to [4, 5].

Here, we will study the following two sets of bounded linear operators.

Definition 1.1. Let X, Y be Banach spaces.

(i) $\mathcal{A}_{\|\cdot\|}(X, Y)$ stands for the set of all norm attaining operators $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies
$$\|T(x)\| > 1 - \eta(\varepsilon, T),$$
there is $x_0 \in S_X$ such that
$$\|T(x_0)\| = \|T\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

(ii) $\mathcal{A}_{\mathrm{nu}}(X)$ stands for the set of all numerical radius attaining operators $T \in \mathcal{L}(X)$ with $\nu(T) = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon, T) > 0$ such that whenever $(x, x^*) \in \Pi(X)$ satisfies
$$|\langle x^*, T(x) \rangle| > 1 - \eta(\varepsilon, T),$$
there is $(x_0, x_0^*) \in \Pi(X)$ such that
$$|\langle x_0^*, T(x_0) \rangle| = \nu(T) = 1, \quad \|x_0 - x\| < \varepsilon, \quad \text{and} \quad \|x_0^* - x^*\| < \varepsilon.$$

In order to show that $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$ (respectively, $\mathcal{A}_{\mathrm{nu}}(X)$), we have to show first that T attains the norm (respectively, the numerical radius) and, as we already mentioned, we do not have to assume that X is reflexive anymore as in the **BPBop**. Obviously, any norm one operator (respectively, numerical radius one operator) which does not attain the norm (respectively, the numerical radius) cannot belong to $\mathcal{A}_{\|\cdot\|}(X, Y)$ (respectively, to $\mathcal{A}_{\mathrm{nu}}(X)$). For that reason, most of the concrete operators defined in the paper are norm one (respectively, numerical radius one) and attain the norm (respectively, the numerical radius).

Let us describe the contents of this paper. In the next section, we show that, as expected, when we are working with finite dimensional spaces, we have a positive result. That is, if $\dim(X) < \infty$, then the set $\mathcal{A}_{\|\cdot\|}(X, Y)$ coincides with the sphere of $\mathcal{L}(X, Y)$ for every Banach space Y and the set $\mathcal{A}_{\mathrm{nu}}(X)$ coincides with the set of all operators with numerical radius one. As a consequence of it, we get that every norm one functional on c_0 which attains the norm belongs to $\mathcal{A}_{\|\cdot\|}(c_0, \mathbb{K})$ by using the canonical embedding from a finite dimensional Euclidian space $(\mathbb{K}^n, \|\cdot\|_\infty)$ (the space $\mathbb{K}^n$ with the topology induced by the norm of c_0) into c_0. We notice that the same result is also true for norm one functionals on ℓ_p, when $1 < p < \infty$, or on Hilbert spaces as a consequence of a result in [11]. On the other hand, we present examples of norm one functionals on ℓ_1 and ℓ_∞ which attain the norm but cannot be in $\mathcal{A}_{\|\cdot\|}(\ell_1, \mathbb{K})$ and $\mathcal{A}_{\|\cdot\|}(\ell_\infty, \mathbb{K})$, respectively. Next, we show that under some assumptions on the Banach space X, the sphere of the compact operators is contained in $\mathcal{A}_{\|\cdot\|}(X, Y)$ and also the set of all compact operators T with $\nu(T) = \|T\| = 1$ belongs to $\mathcal{A}_{\mathrm{nu}}(X)$. We present some counterexamples that show that this is no longer true by dropping some of these hypothesis. We also present some conditions on X that give the possibility to pass from $\mathcal{A}_{\|\cdot\|}(X, Y)$ to $\mathcal{A}_{\|\cdot\|}(Y^*, X^*)$ (analogously, from $\mathcal{A}_{\mathrm{nu}}(X)$ to $\mathcal{A}_{\mathrm{nu}}(X^*)$). By observing some examples, we show that this duality is not true in general. Moreover, we prove that if P_N is the natural projection on c_0 or ℓ_p with $1 \leqslant p < \infty$, then $P_N \in \mathcal{A}_{\|\cdot\|}(X, X) \cap \mathcal{A}_{\mathrm{nu}}(X)$. Finally, in the last section, we connect these sets by using direct sums and proving that these results are no longer true for general absolute sums.

2. The results

Recall that in finite dimensional Banach spaces, every operator T attains the norm and numerical radius by compactness. The following result shows that, when X is finite dimensional, the classes $\mathcal{A}_{\|\cdot\|}(X, Y)$ and $\mathcal{A}_{\mathrm{nu}}(X)$ are the sets of those operators which have norm one and numerical radius one, respectively.

Theorem 2.1. *Let X be a finite dimensional Banach space. Then*

(i) $\mathcal{A}_{\|\cdot\|}(X, Y) = \{T \in \mathcal{L}(X, Y) : \|T\| = 1\}$ for any Banach space Y,
(ii) $\mathcal{A}_{\mathrm{nu}}(X) = \{T \in \mathcal{L}(X) : \nu(T) = 1\}$,
(iii) Every norm one functional on c_0 which attains the norm belongs to $\mathcal{A}_{\|\cdot\|}(c_0, \mathbb{K})$.

Proof. Both items (i) and (ii) are proved by contradiction. We show (ii). Notice that we just have to prove that $\{T \in \mathcal{L}(X) : \nu(T) = 1\} \subset \mathcal{A}_{\mathrm{nu}}(X)$. If this is not the case, then there are $\varepsilon_0 > 0$ and a numerical radius attaining operator $T \in \mathcal{L}(X)$ with $\nu(T) = 1$ such that for all $n \in \mathbb{N}$, there is $(x_n, x_n^*) \in \Pi(X)$ with

$$1 \geqslant |\langle x_n^*, T(x_n)\rangle| \geqslant 1 - \frac{1}{n}$$

and whenever $(x, x^*) \in \Pi(X)$ satisfies $\|x - x_n\| < \varepsilon_0$ and $\|x^* - x_n^*\| < \varepsilon_0$, we have $|\langle x^*, T(x)\rangle| < 1$. By compactness, there are subsequences of (x_n) and (x_n^*) which we denote again by (x_n) and (x_n^*), and there are $x_0 \in B_X$ and $x_0^* \in B_{X^*}$ such that $x_n \longrightarrow x_0$ and $x_n^* \longrightarrow x_0^*$ as $n \to \infty$. Since $\langle x_n^*, x_n\rangle = 1$ for every $n \in \mathbb{N}$, we have that $\langle x_0^*, x_0\rangle = 1$. This shows that $(x_0, x_0^*) \in \Pi(X)$. Now, since $|\langle x_n^*, T(x_n)\rangle| \longrightarrow 1$, we get that $|\langle x_0^*, T(x_0)\rangle| = 1$, which is a contradiction.

To prove (iii), suppose $x^* \in S_{c_0^*}$ attains its norm at some point in B_{c_0}. We can choose $n_0 \in \mathbb{N}$ so that $x^*(n) = 0$ for every $n > n_0$. Let $\Psi : (\mathbb{K}^{n_0}, \|\cdot\|_\infty) \to c_0$ be the canonical embedding into c_0 that sends $(k_1, \cdots, k_{n_0}) \mapsto (k_1, \cdots, k_{n_0}, 0, 0, \cdots)$. It is easy to see that $\|\Psi\| = 1$. Moreover, $\|x^* \circ \Psi\| = 1$, so (i) implies that $x^* \circ \Psi \in \mathcal{A}_{\|\cdot\|}(\mathbb{K}^{n_0}, \mathbb{K})$. Given $\varepsilon > 0$, suppose that $|\langle x^*, x_0\rangle| > 1 - \eta(\varepsilon, x^* \circ \Psi)$ for some point $x_0 \in S_{c_0}$. Then $|(x^* \circ \Psi)(\Psi^{-1}(x_0))| > 1 - \eta(\varepsilon, x^* \circ \Psi)$, so there exists $u \in S_{\mathbb{K}^{n_0}}$ such that $|(x^* \circ \Psi)(u)| = 1$ and $\|u - \Psi^{-1}(x_0)\| < \varepsilon$. It follows that x^* attains its norm at $\Psi(u) \in B_{c_0}$ and $\|\Psi(u) - x_0\| < \varepsilon$.

□

On the other hand, concerning ℓ_p-spaces for $1 \leqslant p \leqslant \infty$, we have the following result. Recall that every uniformly convex space is reflexive and that every bounded linear functional defined on a reflexive space attains the norm. For simplicity, we denote by $\mathrm{NA}(X)$ the set of norm attaining functionals on a Banach space X.

Proposition 2.2. *Let X be a Banach space.*

(i) If X is uniformly convex, then $S_{X^*} \subset \mathcal{A}_{\|\cdot\|}(X,\mathbb{K})$.
(ii) There is $x^* \in \mathrm{NA}(\ell_1) \cap S_{\ell_1^*}$ such that $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_1,\mathbb{K})$.
(iii) There is $x^* \in \mathrm{NA}(\ell_\infty) \cap S_{\ell_\infty^*}$ such that $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty,\mathbb{K})$.

Proof. (i). If X is a uniformly convex Banach space, then by [11, Theorem 2.1], every norm one functional on X belongs to the set $\mathcal{A}_{\|\cdot\|}(X,\mathbb{K})$. In particular, every norm one functional on a Hilbert space H or ℓ_p $(1<p<\infty)$ belongs to $\mathcal{A}_{\|\cdot\|}(H,\mathbb{K})$ and $\mathcal{A}_{\|\cdot\|}(\ell_p,\mathbb{K})$, respectively.

(ii). Indeed, consider the norm one functional $z^* := \left(1, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n}{n+1}, \ldots\right) \in \ell_\infty$. Notice that z^* is a norm attaining functional and the rotations of $e_1 \in S_{\ell_1}$ are the only norming points of z^*. This is so because if $|\langle z^*, z\rangle| = 1$ with $z \in S_{\ell_1}$, then

$$1 = |\langle z^*, z\rangle| = \left| z(1) + \sum_{n=2}^{\infty} \frac{n-1}{n} z(n)\right| \leqslant |z(1)| + \sum_{n=2}^{\infty} \frac{n-1}{n}|z(n)| \leqslant |z(1)| + \sum_{n=2}^{\infty} |z(n)| = \|z\|_1 = 1.$$

So, $\sum_{n\geqslant 2}\left(1-\frac{n-1}{n}\right)|z(n)| = 0$. This implies that $z(n)=0$ for $n \geqslant 2$ and then z is of the form $z = e^{i\theta}e_1$ for some θ.

Given $\varepsilon > 0$, suppose that there is such a $\eta(\varepsilon, z^*) > 0$. We take $k \in \mathbb{N}$ to be such that $\frac{1}{k} < \eta(\varepsilon, z^*)$ and then

$$|\langle z^*, e_k\rangle| = 1 - \frac{1}{k} > 1 - \eta(\varepsilon, z^*),$$

This means that there is $z \in S_{\ell_1}$ such that $|\langle z^*, z\rangle| = 1$ and $\|z - e_k\|_1 < \varepsilon$. By the first part $z = e^{i\theta}e_1$ but $\|e^{i\theta}e_1 - e_k\|_1 = 2$, which is a contradiction. So, z^* cannot belong to $\mathcal{A}_{\|\cdot\|}(\ell_1,\mathbb{K})$ although z^* attains its norm.

(iii). Consider the functional $x^* := \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right)$, viewing ℓ_∞ as a real space. We see x^* as an element in S_{ℓ_1}, which is embedded in $S_{\ell_\infty^*}$. We claim that the only elements where x^* attains the norm are either $(1,1,1,\ldots) \in S_{\ell_\infty}$ or $(-1,-1,-1,\ldots) \in S_{\ell_\infty}$. Indeed, denote $x^* = (x^*(n))_{n=1}^\infty$ and suppose that there is $z = (z(n))_{n=1}^\infty \in S_{\ell_\infty}$ such that $|\langle x^*, z\rangle| = \|x^*\| = 1$. Then,

$$1 = |\langle x^*, z\rangle| = \left|\sum_{n=1}^{\infty} x^*(n) z(n)\right| \leqslant \sum_{n=1}^{\infty} |x^*(n)||z(n)| \leqslant \sum_{n=1}^{\infty} |x^*(n)| = 1.$$

So, $|z(n)| = 1$ for every $n \in \mathbb{N}$. First assume that there is $n_0 \geqslant 1$ such that $z(n_0) = -1$ (so, we can pick n_0 to be a minimal element). If $n_0 > 1$, then

$$\begin{aligned} |\langle x^*, z\rangle| &= \left|\sum_{n=1}^{n_0-1} \frac{1}{2^n} + \left(-\frac{1}{2^{n_0}}\right) + \sum_{n=n_0+1}^{\infty} \frac{z(n)}{2^n}\right| \\ &= \left|\left(1 - \frac{1}{2^{n_0-1}}\right) - \frac{1}{2^{n_0}} + \sum_{n=n_0+1}^{\infty} \frac{z(n)}{2^n}\right| \leqslant \left(1 - \frac{3}{2^{n_0}}\right) + \frac{1}{2^{n_0}} < 1, \end{aligned}$$

which is a contradiction and means that if $z(1) = 1$ then z should be $(1,1,1,\ldots)$. On the other hand, if $n_0 = 1$, then we consider $n_1 \in \mathbb{N}$ to be the first index such that $z(n_1) = 1$. By arguing at the same way as before, we get again that $|\langle x^*, z\rangle| < 1$.

So, if we assume that such a $\eta(\varepsilon, x^*) > 0$ exists, we take $k \in \mathbb{N}$ with $2^k\eta(\varepsilon, x^*) > 1$, and consider the element $e_1 + \ldots + e_k \in S_{\ell_\infty}$. Then,

$$|\langle x^*, e_1 + \ldots + e_k\rangle| = 1 - \frac{1}{2^k} > 1 - \eta(\varepsilon, x^*).$$

So, there is $x \in S_{\ell_\infty}$ such that $|\langle x^*, x\rangle| = 1$ and $\|x - (e_1 + \ldots + e_k)\|_\infty < \varepsilon$, which leads to a contradiction , since by using the first part, we would have either that $\|x - (e_1 + \ldots + e_k)\|_\infty = \|(1,1,1,\ldots) - (e_1+\ldots+e_k)\|_\infty = 1$ or $\|x-(e_1+\ldots+e_k)\|_\infty = \|(-1,-1,-1,\ldots)-(e_1+\ldots+e_k)\|_\infty = 2$. So, $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty, \mathbb{K})$. $\square$

Let us now consider linear operators instead of functionals. In order to find an operator which does not belong neither to $\mathcal{A}_{\|\cdot\|}(X,Y)$ nor $\mathcal{A}_{\text{nu}}(X)$, it is clear that it is enough to consider any operator which has norm and numerical radius one, but it does not attain neither the norm nor numerical radius. Nevertheless, in Example 2.3, we define an operator which cannot belong neither to $\mathcal{A}_{\|\cdot\|}(X,X)$ nor to $\mathcal{A}_{\text{nu}}(X)$ even though it attains its norm and numerical radius.

Example 2.3. We consider the spaces ℓ_p and ℓ_q as $\ell_p(\ell_p^2)$ and $\ell_q(\ell_q^2)$, respectively, where $\ell_p^2 = (\mathbb{K}^2, \|\cdot\|_p)$. For each $n \in \mathbb{N}$, define $T_n : \ell_p^2 \longrightarrow \ell_p^2$ by

$$T_n(x,y) := \left(\left(1 - \frac{1}{2n}\right)x, y\right) \qquad ((x,y) \in \ell_p^2).$$

Now, define $T : \ell_p \longrightarrow \ell_p$ as

$$T(z) := (T_n(x(n), y(n)))_n = \left(\left(1 - \frac{1}{2n}\right)x(n), y(n)\right)_n \qquad (z = ((x(n), y(n)))_n \in \ell_p).$$

In [7, Theorem 2.21.(ii)] it was showed that the operator T attains the norm but $T \notin \mathcal{A}_{\|\cdot\|}(\ell_p, \ell_p)$.

Let us also see that $T \notin \mathcal{A}_{\text{nu}}(X)$. Let e_i^2 be the unit canonical vectors of ℓ_p^2 and ℓ_q^2 for $i = 1,2$, that is, $e_1^2 = (1,0)$ and $e_2^2 = (0,1)$. Consider $e_{i,n} := ((0,0),\ldots,(0,0),\underbrace{e_i^2}_{n\text{-th}},(0,0),\ldots) \in S_{\ell_p}$ and $e_{i,n}^* := ((0,0),\ldots,(0,0),\underbrace{e_i^2}_{n\text{-th}},(0,0),\ldots) \in S_{\ell_q}$ for $i = 1,2$. Since $|\langle e_{2,n}^*, T(e_{2,n})\rangle| = 1$, T attains its numerical radius and $\nu(T) = \|T\| = 1$. Suppose that $T \in \mathcal{A}_{\text{nu}}(\ell_p)$ and consider $\frac{1}{2n} < \eta(\varepsilon, T)$ for a given $\varepsilon \in (0,1)$. Since $\nu(T) = \|e_{1,n}\|_p = \|e_{1,n}^*\|_q = \langle e_{1,n}^*, e_{1,n}\rangle = 1$ and

$$|\langle e_{1,n}^*, T(e_{1,n})\rangle| = 1 - \frac{1}{2n} > 1 - \eta(\varepsilon, T),$$

there is $(w, w^*) \in \Pi(\ell_p)$ such that $|\langle w^*, T(w)\rangle| = 1$, $\|w - e_{1,n}\|_p < \varepsilon$ and $\|w^* - e_{1,n}^*\|_q < \varepsilon$. Since $\|T\| = 1$ and $|\langle w^*, T(w)\rangle| = 1$, it follows that $\|T(w)\|_p = 1$. If we denote $w = ((u(n), v(n)))_n \in S_{\ell_p}$, then it is possible to show that $u(j) = 0$ for all $j \in \mathbb{N}$. This implies that $\|w - e_{1,n}\|_p = \|((0, v(n)))_n - e_{1,n}\|_p \geqslant 1 > \varepsilon$, which is a contradiction.

Proposition 2.4. *Let X be a Banach space.*

(i) Every isometry on X belongs to $\mathcal{A}_{\|\cdot\|}(X,X)$.
(ii) There is an isometry on $X = \ell_2$ such that it does not belong to $\mathcal{A}_{\text{nu}}(X)$.

Proof. Notice that it is immediate to prove that every isometry on a Banach space X belongs to the set $\mathcal{A}_{\|\cdot\|}(X,X)$. On the other hand, this does not hold for the set $\mathcal{A}_{\text{nu}}(X)$. Consider the right shift operator $R : \ell_2 \to \ell_2$, that is, $R(x(1), x(2), x(3), \cdots) = (0, x(1), x(2), \cdots)$ for every $x \in \ell_2$. It is known that the numerical range $W(R)$ of R is the open unit disk $\mathbb{D}$ in the complex plane (see, for example, [13, Example 2]) which implies that $\nu(R) = 1$, but $|\langle Rx, x\rangle| < 1$ for every $x \in S_{\ell_2}$. $\square$

Recall that a Banach space X satisfies the Kadec-Klee property when the weak and norm topologies coincide on the unit sphere S_X. It is well known that every locally uniformly rotund space (LUR, for short) satisfies the Kadec-Klee property (the converse is not true, e.g., ℓ_1^2). Recall also that, by the Šmulian lemma, the norm of X is Fréchet differentiable at x if and only if $(x_n^*) \subset S_{X^*}$ is convergent whenever $\lim_n \langle x_n^*, x\rangle = 1$. In the next result, under some assumptions on the involved Banach spaces, we show that some subsets of the space of all compact operators belong to the classes $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$.

Theorem 2.5. *Let X be a reflexive space which satisfies the Kadec-Klee property. Then,*

(i) $S_{\mathcal{K}(X,Y)} \subset \mathcal{A}_{\|\cdot\|}(X,Y)$ *for every Banach space Y.*

(ii) $\{T \in \mathcal{K}(X) : \nu(T) = \|T\| = 1\} \subset \mathcal{A}_{\mathrm{nu}}(X)$ *whenever X is Fréchet differentiable.*

Proof. We prove (ii). Suppose by contradiction that it is not true. Then, there are $\varepsilon_0 \in (0,1)$ and a compact operator $T \in \mathcal{K}(X)$ with $\nu(T) = \|T\| = 1$ such that for every $n \in \mathbb{N}$, there is $(x_n, x_n^*) \in \Pi(X)$ such that

$$1 \geqslant |\langle x_n^*, T(x_n)\rangle| \geqslant 1 - \frac{1}{n} \tag{1}$$

and whenever $(x, x^*) \in \Pi(X)$ satisfies $\|x - x_n\| < \varepsilon_0$ and $\|x^* - x_n^*\| < \varepsilon_0$, we have $|\langle x^*, T(x)\rangle| < 1$.

Since X is reflexive, B_X is weakly compact and then, by Eberlein-Šmulian theorem, there are a subsequence of (x_n), which we denote again by (x_n), and $x_0 \in B_X$ such that $x_n \xrightarrow{w} x_0$. Since T is completely continuous, we have that $T(x_n) \longrightarrow T(x_0)$ in norm. From this and

$$1 = \nu(T) = \|T\| \geqslant \|T(x_n)\| \geqslant |\langle x_n^*, T(x_n)\rangle| \longrightarrow 1,$$

we get that $\|T(x_0)\| = 1$. This shows that $x_0 \in S_X$. Since w and norm topologies coincide in S_X, we have that $x_n \longrightarrow x_0$ in norm. Notice now that for each $n \in \mathbb{N}$, we have

$$\begin{aligned} 1 \geqslant |\langle x_n^*, T(x_0)\rangle| &= |\langle x_n^*, T(x_n)\rangle| + |\langle x_n^*, T(x_0 - x_n)\rangle| \\ &\geqslant |\langle x_n^*, T(x_n)\rangle| - \|x_0 - x_n\|. \end{aligned}$$

Since x_n converges to x_0 in norm, by using (1), we get that $|\langle x_n^*, T(x_0)\rangle| \longrightarrow 1$.

By compactness of $B_{\mathbb{C}}$, there are a subsequence of (x_n^*), which we denote again by (x_n^*), and some $\theta \in [0, 2\pi)$ such that $\langle x_n^*, T(x_0)\rangle$ converges to $e^{i\theta}$. Let $S \in \mathcal{K}(X)$ be the operator defined by $S := e^{-i\theta} \cdot T$. One clearly has that $S(x_0) \in S_X$ and $\langle x_n^*, S(x_0)\rangle$ converges to 1. On account of the Fréchet differentiability of X, by Šmulian lemma, there is $x_0^* \in B_{X^*}$ such that $x_n^* \longrightarrow x_0^*$ in norm. Since $\langle x_n^*, x_n\rangle = 1$ for every $n \in \mathbb{N}$, we get that $\langle x_0^*, x_0\rangle = 1$. So $x_0^* \in S_{X^*}$ and so $(x_0, x_0^*) \in \Pi(X)$. Finally, in view of (1) and $|\langle x_n^*, T(x_n)\rangle| \longrightarrow |\langle x_0^*, T(x_0)\rangle|$, we get that $|\langle x_0^*, T(x_0)\rangle| = 1$. This is a contradiction.

Note that (i) is obtained by using a very similar argument as above (see [15]). □

In fact, the above argument shows, under the same assumptions on (ii), that every compact operator T which has norm and numerical radius 1 attains its numerical radius. Notice also that the identity operator always belongs to $\mathcal{A}_{\mathrm{nu}}(X)$ whereas it is not compact unless X is finite dimensional. So, in the infinite dimensional setting, the inclusion in Theorem 2.5.(ii) must be strict. On the other hand, since every operator from a reflexive space into a space which satisfies the Schur's property is compact and Hilbert spaces satisfy all the hypothesis from Theorem 2.5, we have the following consequence.

Corollary 2.6. *Let X be a reflexive Banach space with the Kadec-Klee property and let H be a Hilbert space.*

(i) If Y has the Schur property, then $\mathcal{A}_{\|\cdot\|}(X,Y) = S_{\mathcal{L}(X,Y)}$.

(ii) If $T \in \mathcal{K}(H)$ is with $\nu(T) = \|T\| = 1$, then $T \in \mathcal{A}_{\mathrm{nu}}(H)$.

Remark 2.7. We notice that there is a norm one compact operator T on ℓ_1 (which is non reflexive and has Schur property) such that $T \notin \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$. Indeed, if $T : \ell_1 \to \ell_1$ is defined as

$$T(x) = \left(\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x(n)\right) e_1 \quad (x \in \ell_1)$$

then $\|T\| = 1$ but T never attains its norm. On the other hand, in [2, Example 1.8], the author constructed a non compact operator S on a separable Hilbert space such that $\nu(S) = \|S\| = 1$ but does not attains its numerical radius. So, this S cannot belong to $\mathcal{A}_{\mathrm{nu}}(H)$.

In the next example, we present a numerical radius attaining compact operator $S \notin \mathcal{A}_{\text{nu}}$ with $\nu(S) = \|S\| = 1$ defined on a Banach space X which is not reflexive, its norm is nowhere Fréchet differentiable and satisfies the Schur's property (and, in particular, the Kadec-Klee property).

Example 2.8. Let c_0 be the real space of all sequences which converge to zero. Consider the operator $T : c_0 \longrightarrow c_0$ to be defined as

$$(T(x))(1) = \sum_{j=1}^{\infty} \frac{1}{2^j} x(j) \quad \text{and} \quad (T(x))(k) = 0 \quad (k \geqslant 2) \quad (x = (x(j))_{j=1}^{\infty} \in c_0). \tag{2}$$

It is proved in [2, Proposition 2.8] that $\|T\| = \nu(T) = 1$ but T does not attain neither the norm nor the numerical radius. In particular, T cannot belong neither to $\mathcal{A}_{\|\cdot\|}(c_0, c_0)$ nor to $\mathcal{A}_{\text{nu}}(c_0)$. We claim that $S = T^*$ is an operator which is compact numerical radius attaining operator with $\nu(S) = \|S\| = 1$ but does not belong to $\mathcal{A}_{\text{nu}}$. Notice that $S : \ell_1 \longrightarrow \ell_1$ is given by

$$S(y) = \sum_{j=1}^{\infty} \frac{y(1)}{2^j} e_j \quad (y = (y(j))_{j=1}^{\infty} \in \ell_1).$$

Moreover, notice also that $\nu(S) = \nu(T) = 1$, $\langle z, e_1 \rangle = 1$ where $z = (1, 1, 1, \dots) \in S_{\ell_\infty}$, and that

$$\langle z, Se_1 \rangle = \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$$

which implies that S attains the numerical radius (and the norm).

Let us first notice that $S \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$. Indeed, given $\varepsilon > 0$, take $x \in S_{\ell_1}$ such that $\|S(x)\|_1 > 1 - \frac{\varepsilon}{2}$, that is,

$$\sum_{j=1}^{\infty} \frac{|x(1)|}{2^j} > 1 - \frac{\varepsilon}{2}.$$

Thus, $|x(1)| > 1 - \frac{\varepsilon}{2}$ and $\sum_{j=2}^{\infty} |x(j)| \leqslant \frac{\varepsilon}{2}$. Consider $y = \left(\frac{x(1)}{|x(1)|}, 0, 0, \dots \right) \in S_{\ell_1}$, then

$$\|S(y)\|_1 = \sum_{j=1}^{\infty} \left(\left| \frac{x(1)}{|x(1)|} \right| \frac{1}{2^j} \right) = 1$$

and

$$\|x - y\|_1 = |x(1) - y(1)| + \sum_{j=2}^{\infty} |x(j)| \leqslant (1 - |x(1)|) + \frac{\varepsilon}{2} < \varepsilon.$$

This shows that $S \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$.

On the other hand, S cannot be in $\mathcal{A}_{\text{nu}}(\ell_1)$. Indeed, observe that if $(y, z) \in \Pi(\ell_1)$ satisfy $|\langle z, S(y) \rangle| = 1$, then

$$\sum_{j=1}^{\infty} |y(j)| = 1, \quad \sum_{j=1}^{\infty} y(j) z(j) = 1, \quad \left| \sum_{j=1}^{\infty} \frac{1}{2^j} y(1) z(j) \right| = 1, \quad \text{and} \quad \max_{j \in \mathbb{N}} |z(j)| = 1.$$

From the third equality, we have

$$1 = \left| \sum_{j=1}^{\infty} \frac{1}{2^j} y(1) z(j) \right| \leqslant |y(1)| \sum_{j=1}^{\infty} \frac{1}{2^j} = |y(1)| \leqslant 1.$$

This implies that $|y(1)| = 1$ and $|y(j)| = 0$ for $j \neq 1$ because of (1) above since $\|y\|_1 = 1$. Then, from the second equality, we get that $y(1)z(1) = 1$, that is, $z(1) = y(1) = \pm 1$. Then, by using the third equality again,

$$1 = \left| \frac{y(1)z(1)}{2} + \sum_{j=2}^{\infty} \frac{y(1)}{2^j} z(j) \right| = \left| \frac{1}{2} + \sum_{j=2}^{\infty} \frac{y(1)}{2^j} z(j) \right|.$$

So,

$$\left(\frac{1}{2}+\sum_{j=2}^{\infty}\frac{y(1)}{2^j}z(j)\right)^2 = 1 = \left(\frac{1}{2}+\sum_{j=2}^{\infty}\frac{1}{2^j}\right)^2$$

and then,

$$\sum_{j=2}^{\infty}\frac{y(1)}{2^j}z(j) = \sum_{j=2}^{\infty}\frac{1}{2^j} \Longrightarrow \sum_{j=2}^{\infty}\frac{1}{2^j}(1-y(1)z(j)) = 0.$$

So, $y(1)z(j) = 1$ for all $j \geqslant 2$ and then $z(j) = y(1)$ for all $j \geqslant 2$. Therefore, the only possible candidates are $y = (1,0,0,0,\ldots)$ and $z = (1,1,1,1,\ldots)$ or $y = (-1,0,0,0,\ldots)$ and $z = (-1,-1,-1,-1,\ldots)$.

Suppose, by contradiction, that for a given $\varepsilon \in (0,1)$, there is $\eta(\varepsilon, S) > 0$. Let $n_0 \in \mathbb{N}$ be such that

$$\sum_{j=1}^{n_0}\frac{1}{2^j} > 1-\eta(\varepsilon,S).$$

Set

$$y_0 := (1,0,0,\ldots) \in S_{\ell_1} \quad \text{and} \quad z_0 := (1,1,\ldots,1,\underbrace{1}_{n_0\text{-th}},0,0,\ldots) \in S_{\ell_\infty}.$$

Then, $(y_0, z_0) \in \Pi(\ell_1)$. Note also that

$$|\langle z_0, S(y_0)\rangle| = \left|\sum_{j=1}^{n_0}\frac{y_0(1)}{2^j}z_0(j)\right| = \sum_{j=1}^{n_0}\frac{1}{2^j} > 1-\eta(\varepsilon,S).$$

So, there is $(y,z) \in \Pi(\ell_1)$ such that

$$|\langle z, S(y)\rangle| = 1, \quad \|y-y_0\|_1 < \varepsilon, \quad \text{and} \quad \|z-z_0\|_\infty < \varepsilon.$$

But this is not possible since

$$\varepsilon > \|z-z_0\|_\infty \geqslant |z(n_0+1)-z_0(n_0+1)| \geqslant 1,$$

which is a contradiction. This shows that $S \notin \mathcal{A}_{\text{nu}}(\ell_1)$.

Let us recall that in Corollary 2.6, we proved that if a compact operator T defined on a Hilbert space is such that $\nu(T) = \|T\| = 1$, then T must belong to the set $\mathcal{A}_{\text{nu}}(H)$. However, the following example shows that it can happen that $1 = \nu(T) < \|T\|$ and even so $T \in \mathcal{A}_{\text{nu}}(H)$.

Example 2.9. Let H be a separable infinite dimensional real Hilbert space. We will define an operator T on H such that T is compact, attains its numerical radius, $1 = \nu(T) < \|T\|$ and $T \in \mathcal{A}_{\text{nu}}(H)$. Let $0 < \alpha \leqslant 1$ and $\{\alpha_n\}$ be a sequence such that $\alpha_1 > 1$, $0 < \alpha_n < 1$ for $n \geqslant 2$, and $\alpha_n \to 0$ as $n \to \infty$. Let $\{J_1, J_2, J_3\}$ be a partition of $\mathbb{N}$ such that $|J_1| = |J_2| = \aleph_0$, $|J_3| = \ell < \infty$. Write the subsets J_1, J_2 as $J_1 = \{n_k : k \geqslant 1\}$, $J_2 = \{m_k : k \geqslant 1\}$ where $n_1 \leqslant n_2 \leqslant \ldots$, $m_1 \leqslant m_2 \leqslant \ldots$ and each n_k corresponds to m_k via an one-to-one correspondence between J_1 and J_2. Define $T : H \to H$ by

$$T(e_{n_k}) = -\alpha_k e_{m_k} \ (k \in \mathbb{N}), \quad T(e_{m_k}) = \alpha_k e_{n_k} \ (k \in \mathbb{N}), \quad T(e_n) = \alpha e_n \ (n \in J_3),$$

where $\{e_n : n \geqslant 1\}$ is an orthonormal basis of H.

We claim that T is a compact operator. Indeed, note first that for every $x \in H$, we have

$$T(x) = \sum_{n=1}^{\infty}\langle x, e_n\rangle T(e_n) = \sum_{\substack{k\in\mathbb{N}\\ n\in J_3}} -\alpha_k\langle x, e_{n_k}\rangle e_{m_k} + \alpha_k\langle x, e_{m_k}\rangle e_{n_k} + \alpha\langle x, e_n\rangle e_n. \tag{3}$$

Now since J_3 is finite, we may take j sufficiently large so that $j \notin J_3$ and $j \in J_1 \cup J_2$. Assume $j \in J_1$. If $j = n_k$ for some $k \geqslant 1$, then $\|Te_j\| = |\alpha_k|$. Since $\alpha_n \to 0$ as $n \to \infty$, we have $\|Te_j\| \to 0$ as $j \to \infty$. For a given $\varepsilon > 0$, choose j_0 such that $\|Te_j\| < \varepsilon$ for $j \geqslant j_0$. For each $n \geqslant 1$, set

$T_n := \sum_{j=1}^{n} \langle \cdot e_j \rangle T(e_j)$. Let us observe that for every $x \in S_H$ and $n \geqslant m \geqslant j_0$ (j_0 is large enough so that $j_0 \notin J_3$), we have that

$$\|(T_n - T_m)(x)\| = \left\| \sum_{j=m}^{n} \langle x, e_j \rangle T(e_j) \right\| = \left(\sum_{j=m}^{n} \|\langle x, e_j \rangle T(e_j)\|^2 \right)^{1/2} < \varepsilon \left(\sum_{j=m}^{n} |\langle x, e_j \rangle|^2 \right)^{1/2} \leqslant \varepsilon$$

since $\{Te_j\}_{j=m}^{n}$ are orthogonal. This shows that (T_n) is a Cauchy sequence in $\mathcal{L}(H)$ which converges to T. Since each T_n is finite-rank, it follows that T is compact.

Now we calculate the norm and numerical radius of T. Note first that, by using (3), we have the following equalities:

(i) $\langle T(x), e_{n_k} \rangle = \alpha_k \langle x, e_{m_k} \rangle$,

(ii) $\langle T(x), e_{m_k} \rangle = -\alpha_k \langle x, e_{n_k} \rangle$ and

(iii) $\langle T(x), e_n \rangle = \alpha \langle x, e_n \rangle$,

for $n_k \in J_1$, $m_k \in J_2$ and $n \in J_3$. Then, for each $x \in S_H$, we have

$$\begin{aligned} \langle T(x), x \rangle &= \sum_{n=1}^{\infty} \langle e_n, x \rangle \langle T(x), e_n \rangle \\ &= \sum_{\substack{k \in \mathbb{N} \\ n \in J_3}} \alpha_k \langle e_{n_k}, x \rangle \langle x, e_{m_k} \rangle - \alpha_k \langle e_{m_k}, x \rangle \langle x, e_{n_k} \rangle + \alpha \langle e_n, x \rangle \langle x, e_n \rangle. \end{aligned}$$

The first two terms are canceled out because H is real and then

$$\langle T(x), x \rangle = \alpha \sum_{n \in J_3} |\langle x, e_n \rangle|^2 \tag{4}$$

for $x \in S_H$ which implies that $\nu(T) \leqslant \alpha$. Since $|\langle Te_n, e_n \rangle| = \alpha$ for every $n \in J_3$, we have that T attains its numerical radius and $\nu(T) = \alpha$.

On the other hand, let us notice that, for every $x \in H$, we have

$$\begin{aligned} \|T(x)\|^2 = \sum_{j=1}^{\infty} |\langle T(x), e_j \rangle|^2 &= \sum_{\substack{k \in \mathbb{N} \\ n \in J_3}} |\alpha_k \langle x, e_{m_k} \rangle|^2 + |\alpha_k \langle x, e_{n_k} \rangle|^2 + |\alpha \langle x, e_n \rangle|^2 \\ &\leqslant (\max\{\|\{\alpha_n\}\|_\infty, |\alpha|\})^2 \|x\|^2. \end{aligned} \tag{5}$$

It follows that $\|T\| \leqslant \max\{\|\{\alpha_n\}\|_\infty, |\alpha|\}$. However, we also have

$$\|T\| \geqslant \sup\{\|T(e_n)\| : n \geqslant 1\} = \sup\{|\alpha_k|, |\alpha| : k \geqslant 1\} = \max\{\|\{\alpha_n\}\|_\infty, |\alpha|\}. \tag{6}$$

From (5) and (6), we have $\|T\| = \max\{\|\{\alpha_n\}\|_\infty, |\alpha|\}$. In particular, since $\alpha_1 > 1$, we have $\|T\| > 1 \geqslant \alpha = \nu(T)$.

Now we prove that $T \in \mathcal{A}_{\text{nu}}(H)$ when $\alpha = 1$. For a given $\varepsilon \in (0,1)$, set

$$\eta(\varepsilon, T) := \frac{\varepsilon^2}{4} > 0.$$

and let $x_0 \in S_H$ be such that

$$|\langle T(x_0), x_0 \rangle| > 1 - \frac{\varepsilon^2}{4}.$$

By equation (4), we have that

$$\sum_{n \in J_3} |\langle x_0, e_n \rangle|^2 = |\langle T(x_0), x_0 \rangle| > 1 - \frac{\varepsilon^2}{4}.$$

Hence

$$\sum_{k \in J_1 \cup J_2} |\langle x_0, e_k \rangle|^2 < \frac{\varepsilon^2}{4}.$$

Let π_3 be the projection of H onto the closed subspace $H_3 = \operatorname{span}\{e_n : n \in J_3\}$. Then we have $\pi_3(x_0) = \sum_{n\in J_3}\langle x_0, e_n\rangle e_n$ and

$$\langle T(\pi_3(x_0)), \pi_3(x_0)\rangle = \sum_{n\in J_3} |\langle \pi_3(x_0), e_n\rangle|^2 = \sum_{n\in J_3} |\langle x_0, e_n\rangle|^2.$$

It follows that T attains its numerical radius at $\|\pi_3(x_0)\|^{-1}\pi_3(x_0) \in S_H$. Moreover,

$$\begin{aligned}\left\|\frac{\pi_3(x_0)}{\|\pi_3(x_0)\|} - x_0\right\| &\leqslant \left\|\frac{\pi_3(x_0)}{\|\pi_3(x_0)\|} - \pi_3(x_0)\right\| + \|\pi_3(x_0) - x_0\| \\ &\leqslant |1 - \|\pi_3(x_0)\|| + \left(\sum_{k\in J_1\cup J_2} |\langle x_0, e_k\rangle|^2\right)^{1/2} \\ &< |\|x_0\| - \|\pi_3(x_0)\|| + \frac{\varepsilon}{2} \\ &\leqslant \|x_0 - \pi_3(x_0)\| + \frac{\varepsilon}{2} \\ &< \varepsilon.\end{aligned}$$

This shows that for a given $\varepsilon > 0$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x_0 \in S_H$ satisfies $\langle T(x_0), x_0\rangle > 1 - \eta(\varepsilon, T)$, there is $x_1 \in S_H$ such that $\nu(T) = \langle T(x_1), x_1\rangle = 1$ and $\|x_1 - x_0\| < \varepsilon$.

Example 2.9 gives, in particular, a non trivial example of an operator which belongs to $\mathcal{A}_{\mathrm{nu}}$ but not to $\mathcal{A}_{\|\cdot\|}$. On the other hand, notice that we have proved in Example 2.8 that the operator T^* belongs to $\mathcal{A}_{\|\cdot\|}$ but it cannot belong to $\mathcal{A}_{\mathrm{nu}}$ although it attains its numerical radius and $\|T^*\| = \nu(T^*) = 1$. Let us present next a non trivial example of an operator S on an infinite dimensional Banach space such that $S \in \mathcal{A}_{\|\cdot\|} \cap \mathcal{A}_{\mathrm{nu}}$ and $S^* \in \mathcal{A}_{\|\cdot\|} \cap \mathcal{A}_{\mathrm{nu}}$.

Example 2.10. Let c_0 be a real sequence space. Let $S : c_0 \to c_0$ be the diagonal operator defined by

$$S(x) = \left(x(1), \frac{1}{2}x(2), \frac{1}{2^2}x(3), \ldots\right) \quad (x = (x(j))_{j\in\mathbb{N}} \in c_0).$$

Then it is clear that $\|S\| = \nu(S) = 1$ and S attains its norm and numerical radius. To see that $S \in \mathcal{A}_{\|\cdot\|}(c_0, c_0) \cap \mathcal{A}_{\mathrm{nu}}(c_0)$, let $\varepsilon > 0$ be given and suppose that $\|S(x)\| > 1 - \eta(\varepsilon, T)$ for some $x \in S_{c_0}$, where $\eta(\varepsilon, S) := \min\{\varepsilon/4, 1/4\} > 0$. Observe that

$$\|S(x)\| = \sup_{n\in\mathbb{N}} \frac{1}{2^{n-1}}|x(n)| > 1 - \eta(\varepsilon, S)$$

implies that $|x(1)| > 1 - \eta(\varepsilon, S) > 0$. It follows that $y = (|x(1)|^{-1}x(1), x(2), x(3), \cdots) \in S_{c_0}$, $\|y - x\| < \varepsilon$ and $\|S(y)\| = 1$, so $S \in \mathcal{A}_{\|\cdot\|}(c_0, c_0)$.

Next, suppose that $|\langle y, S(x)\rangle| > 1 - \eta(\varepsilon, S)$ for some $(x, y) \in \Pi(c_0)$ with the same $\eta(\varepsilon, S) > 0$. Note that

$$|y(1)| + \frac{1}{2}\left(\sum_{n=2}^{\infty} |y(n)|\right) \geqslant \left|\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} x(n)y(n)\right| = |\langle y, S(x)\rangle| > 1 - \eta(\varepsilon, S);$$

hence $|y(1)| > 1 - 2\eta(\varepsilon, S)$ and $\sum_{n=2}^{\infty} |y(n)| \leqslant 2\eta(\varepsilon, S)$. Since $(x, y) \in \Pi(c_0)$, we also have that

$$x(1)y(1) + 2\eta(\varepsilon, S) \geqslant x(1)y(1) + \sum_{n=2}^{\infty} |x(n)||y(n)| \geqslant \sum_{n=1}^{\infty} x(n)y(n) = 1;$$

thus $x(1)y(1) \geqslant 1 - 2\eta(\varepsilon, S) > 0$. On the other hand,

$$|x(1)| + \frac{1}{2}\left(\sum_{n=2}^{\infty} |y(n)|\right) \geqslant 1 - \eta(\varepsilon, S)$$

which implies that $|x(1)| > 1 - 2\eta(\varepsilon, S)$. Consider $\overline{x} = (|x(1)|^{-1}x(1), x(2), x(3), \cdots) \in S_{c_0}$ and $\overline{y} = (|y(1)|^{-1}y(1), 0, 0, \cdots) \in S_{\ell_1}$. Then $(\overline{x}, \overline{y}) \in \Pi(c_0)$, $\langle \overline{y}, T\overline{x}\rangle = 1$, $\|\overline{x} - x\| < \varepsilon$ and $\|\overline{y} - y\| < \varepsilon$. So, $S \in \mathcal{A}_{\text{nu}}(c_0)$.

By definition, one can see easily that

$$S^*(y) = \left(y(1), \frac{1}{2}y(2), \frac{1}{2^2}y(3), \dots\right) \quad (y = (y(n))_{n\in\mathbb{N}} \in \ell_1)$$

and that $S^* \in \mathcal{A}_{\text{nu}}(\ell_1)$ by using a similar argument than the one we just used. As for $\mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$, given $\varepsilon > 0$, let $x \in S_{\ell_1}$ be such that $\|S^*(x)\| > 1 - \eta(\varepsilon, S^*)$, where $\eta(\varepsilon, S^*) := \min\left\{\frac{\varepsilon}{4}, \frac{1}{4}\right\}$. One gets that

$$1 - \eta(\varepsilon, S^*) < \|S^*(x)\| = |x(1)| + \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{2^{n-2}}|x(n)| \leqslant \frac{1}{2}|x(1)| + \frac{1}{2}\|x\|_1 = \frac{1}{2}|x(1)| + \frac{1}{2}.$$

From here, $|x(1)| > 1 - 2\eta(\varepsilon, S^*)$. Then it is easy to check that the point $y = \left(\frac{x(1)}{|x(1)|}, 0, 0, \dots\right) \in S_{\ell_1}$ satisfies that $\|S^*(y)\| = 1$ and that $\|x - y\| < \varepsilon$, and therefore $S^* \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$.

Observe that it is not true that T^* belongs to $\mathcal{A}_{\|\cdot\|}$ (respectively, $\mathcal{A}_{\text{nu}}$) whenever T belongs to $\mathcal{A}_{\|\cdot\|}$ (respectively, $\mathcal{A}_{\text{nu}}$) in general (see Example 2.8, 2.12 and 2.13). However, if we put some assumptions on the spaces X and Y, then we can obtain the following duality results.

Theorem 2.11. *Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$.*

(i) Suppose that Y be uniformly smooth. If $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$, then $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$.
(ii) Suppose that X be uniformly convex. If $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$, then $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$.
(iii) Suppose that X is reflexive. Then, $T \in \mathcal{A}_{\text{nu}}(X)$ if and only if $T^* \in \mathcal{A}_{\text{nu}}(X^*)$.

Proof. Note that (ii) is just a consequence of (i) since, in this case, X is, in particular, reflexive. Let us prove (i). Let Y be a uniformly smooth Banach space. Let $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$. Then, $\|T^*\| = \|T\| = 1$ and T^* is also norm attaining. In order to prove that $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$, let $\varepsilon \in (0, 1)$ be given and consider $\eta(\varepsilon, T) > 0$. Set

$$\eta(\varepsilon, T^*) := \min\left\{\eta\left(\frac{\delta_{Y^*}(\varepsilon)}{2}, T\right), \frac{\delta_{Y^*}(\varepsilon)}{2}\right\} > 0,$$

where $\varepsilon \mapsto \delta_{Y^*}(\varepsilon)$ stands for the modulus of convexity of Y^*. Pick $y_1^* \in S_{Y^*}$ to satisfy

$$\|T^*(y_1^*)\| > 1 - \eta(\varepsilon, T^*).$$

Since X is reflexive, there is $x_1 \in S_X$ such that

$$\text{Re}\langle y_1^*, T(x_1)\rangle = \text{Re}\langle x_1, T^*(y_1^*)\rangle = \|T^*(y_1^*)\| > 1 - \eta(\varepsilon, T^*).$$

This implies that $\|T(x_1)\| \geqslant \text{Re}\langle y_1^*, T(x_1)\rangle > 1 - \eta(\varepsilon, T^*)$. Since $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$, there is $x_2 \in S_X$ such that

$$\|T(x_2)\| = 1 \quad \text{and} \quad \|x_2 - x_1\| < \frac{\delta_{Y^*}(\varepsilon)}{2}.$$

Take $y_2^* \in S_{Y^*}$ to be such that $\text{Re}\langle y_2^*, T(x_2)\rangle = \|T(x_2)\| = 1$ and notice that

$$\begin{aligned} \text{Re}\langle y_1^*, T(x_2)\rangle &= \text{Re}\langle y_1^*, T(x_1)\rangle + \text{Re}\langle y_1^*, T(x_2 - x_1)\rangle \\ &\geqslant \text{Re}\langle y_1^*, T(x_1)\rangle - \|x_2 - x_1\| \\ &> 1 - \delta_{Y^*}(\varepsilon). \end{aligned}$$

Then,

$$\left\|\frac{y_1^* + y_2^*}{2}\right\| \geqslant \text{Re}\langle \frac{y_1^* + y_2^*}{2}, T(x_2)\rangle > 1 - \delta_{Y^*}(\varepsilon).$$

This shows that $\|y_2^* - y_1^*\| < \varepsilon$. Finally, since

$$1 = \text{Re}\langle y_2^*, T(x_2)\rangle = \text{Re}\langle x_2, T^*(y_2^*)\rangle \leqslant \|T^*(y_2^*)\| \leqslant \|T^*\| = 1,$$

T^* attains its norm at y_2^* which is close to y_1^*. Henceforth, $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$.

Now we prove (iii). Since X is reflexive, we just have to prove one direction. Assume $T \in \mathcal{A}_{\text{nu}}(X)$. Note that $T^* \in \mathcal{L}(X^*)$ also attains its numerical radius. Now let $\varepsilon > 0$ be given and set $\eta(\varepsilon, T^*) := \eta(\varepsilon, T) > 0$. Let $(x_1^*, x_1^{**}) \in \Pi(X^*)$ be such that

$$|\langle x_1^{**}, T^*(x_1^*)\rangle| > 1 - \eta(\varepsilon, T^*).$$

Since X is reflexive, there is $x_1 \in S_X$ such that $x_1 = x_1^{**}$. Then

$$|\langle x_1^*, T(x_1)\rangle| = |\langle x_1, T^*(x_1^*)\rangle| = |\langle x_1^{**}, T^*(x_1^*)\rangle| > 1 - \eta(\varepsilon, T^*) = 1 - \eta(\varepsilon, T).$$

Then there is $(x_2, x_2^*) \in \Pi(X)$ such that

$$|\langle x_2^*, T(x_2)\rangle| = 1, \quad \|x_2 - x_1\| < \varepsilon \quad \text{and} \quad \|x_2^* - x_1^*\| < \varepsilon.$$

But this implies that $|\langle x_2, T^*(x_2^*)\rangle| = \nu(T^*) = 1$ and $\|x_2 - x_1^{**}\| = \|x_2 - x_1\| < \varepsilon$. So $T^* \in \mathcal{A}_{\text{nu}}(X^*)$ as desired. □

In Theorem 2.11, if we drop off the hypothesis on Banach spaces X and Y, it is possible to construct non trivial operators which do not satisfy the conclusion of that result. Recall that, in Example 2.8, we constructed the operator T defined on the non reflexive space c_0 such that $T^* \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$ but $T \notin \mathcal{A}_{\|\cdot\|}(c_0, c_0)$. The next example shows that the adjoint operator T^* of the same T implies the existence of an operator S defined on a non reflexive space X such that $S \in \mathcal{A}_{\|\cdot\|}(X, X)$ but $S^* \notin \mathcal{A}_{\|\cdot\|}(X^*, X^*)$.

Example 2.12. The operator T defined in Example 2.8 is such that $T^{**} \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty, \ell_\infty)$ although $T^* \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$. Indeed, $T^{**} \in \mathcal{L}(\ell_\infty)$ is given by

$$(T^{**}(z))(1) = \sum_{j=1}^{\infty} \frac{1}{2^j} z(j) \quad \text{and} \quad (T^{**}(z))(k) = 0 \ \forall \ k \geqslant 2$$

for $z \in \ell_\infty$. Note that for all $z \in S_{\ell_\infty}$, $\|T^{**}(z)\| = \max_{n \in \mathbb{N}} |T^{**}(z)(n)| \leqslant \sum_{j \in \mathbb{N}} \frac{1}{2^j} |z(j)| \leqslant 1$ and for the vector $u_0 = (1, 1, 1, 1, \ldots) \in S_{\ell_\infty}$, we have $\|T^{**}(u_0)\| = 1 = \|T^{**}\|$. So, $\|T^{**}\| = 1$ and T^{**} attains its norm. Let $z_0 \in S_{\ell_\infty}$ be such that $\|T^{**}(z_0)\|_\infty = 1$. Then,

$$\sup_{n \in \mathbb{N}} |(T^{**}(z_0))(n)| = \left| \sum_{j=1}^{\infty} \frac{1}{2^j} z_0(j) \right| = 1.$$

This shows that $|z_0(j)| = 1$ for all $j \in \mathbb{N}$.

For a given $\varepsilon \in (0, 1)$, suppose that there is $\eta(\varepsilon, T^{**}) > 0$. Let $n_0 \in \mathbb{N}$ be such that $2^n \eta(\varepsilon, T^{**}) > 1$ for every $n \geqslant n_0$. Consider the vector $z \in S_{\ell_\infty}$ defined as $z = (1, 1, \cdots, \underbrace{1}_{n_0\text{-th}}, 0, 0, \cdots)$. Then,

$$\|T^{**}(z)\| = \sum_{j=1}^{n_0} \frac{1}{2^j} = 1 - \frac{1}{2^{n_0}} > 1 - \eta(\varepsilon, T^{**}).$$

However, we have observed that if $\|T^{**}(z_0)\| = 1$ then $|z_0(j)| = 1$ for all $j \in \mathbb{N}$. Thus the vector z cannot be close to norming points of T^{**}. This shows that $T^{**} \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty, \ell_\infty)$.

On the other hand, by changing slightly the definition of the operator in Example 2.8, we define a new operator T on a non reflexive space X such that $T \in \mathcal{A}_{\text{nu}}(X)$ but $T^* \notin \mathcal{A}_{\text{nu}}(X)$.

Example 2.13. Let X be a real c_0 space. Let $T : c_0 \longrightarrow c_0$ be defined as

$$T(x) = \left(\sum_{j=1}^{\infty} \frac{1}{2^j} x(j), x(2), 0, 0, \cdots \right) \quad (x = (x(j))_{j \in \mathbb{N}} \in c_0).$$

We will prove that $T \in \mathcal{A}_{\rm nu}(c_0)$ but $T^* \notin \mathcal{A}_{\rm nu}(\ell_1)$. Indeed, notice that $\nu(T) = \|T\| = 1$ and T attains both the norm and numerical radius. Let $\varepsilon \in (0,1)$ be given and suppose that $|\langle y, T(x)\rangle| > 1 - \eta(\varepsilon, T)$ for some $(x,y) \in \Pi(c_0)$, where $\eta(\varepsilon, T) = \varepsilon/5 > 0$. This means that

$$1 - \eta(\varepsilon, T) < \left| \sum_{j=1}^{\infty} \frac{1}{2^j} x(j)y(1) + x(2)y(2) \right| \leqslant \frac{1}{2}|y(1)| + |x(2)y(2)|$$
$$\leqslant \frac{1}{2}|y(1)| + |y(2)| \leqslant |y(1)| + |y(2)| \leqslant 1.$$

Thus, $|y(1)| < 2\eta(\varepsilon, T)$. Hence, $|x(2)y(2)| > 1 - 2\eta(\varepsilon, T) > 0$ and, in particular, $|x(2)| > 0$ and $|y(2)| > 0$. Also notice that, since $|y(1)| + |y(2)| > 1 - \eta(\varepsilon, T)$, we have that $\sum_{j=3}^{\infty} |y(j)| \leqslant \eta(\varepsilon, T)$. Consider

$$\widetilde{x} = \left(x(1), \frac{x(2)}{|x(2)|}, x(3), x(4) \cdots \right) \in S_{c_0} \quad \text{and} \quad \widetilde{y} = \left(0, \frac{y(2)}{|y(2)|}, 0, 0 \cdots, \right) \in S_{\ell_1}.$$

Then $(\widetilde{x}, \widetilde{y}) \in \Pi(c_0)$, $\|\widetilde{x} - x\|_\infty < \varepsilon$, $\|\widetilde{y} - y\|_1 < \varepsilon$ and $|\langle \widetilde{y}, T\widetilde{x}\rangle| = 1$. This shows that $T \in \mathcal{A}_{\rm nu}(X)$. On the other hand, note that $T^* : \ell_1 \longrightarrow \ell_1$ is given by

$$T^*(y) = \left(\frac{1}{2}y(1), \frac{1}{2^2}y(1) + y(2), \frac{1}{2^3}y(1), \frac{1}{2^4}y(1), \cdots \right), \quad (y = (y(j))_{j\in\mathbb{N}} \in \ell_1).$$

For given $\varepsilon \in (0, 1/2)$, assume that there exists $\eta(\varepsilon, T^*) \in (0, 1/2)$ satisfying that whenever $(y,z) \in \Pi(\ell_1)$ with $|\langle z, T^*y\rangle| > 1-\eta(\varepsilon, T^*)$, there is $(\widetilde{y}, \tilde{z}) \in \Pi(\ell_1)$ such that $|\langle \tilde{z}, T^*\widetilde{y}\rangle| = 1, \|\widetilde{y}-y\| < \varepsilon$, and $\|\tilde{z} - z\| < \varepsilon$. Choose $n_0 \in \mathbb{N}$ so that $2^{n_0+1}\eta(\varepsilon, T^*) > 1$. Note that the pair $(y,z) \in \ell_1 \times \ell_\infty$ defined as

$$y = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \cdots \right) \quad \text{and} \quad z = (1, \cdots, \underbrace{1}_{n_0\text{-th}}, 0, 0 \cdots)$$

satisfies that $(y,z) \in \Pi(\ell_1)$ and

$$|\langle T^*y, z\rangle| = \frac{1}{2}\sum_{j=1}^{n_0} \frac{1}{2^j} + \frac{1}{2} = \frac{1}{2}\left(1 - \frac{1}{2^{n_0}}\right) + \frac{1}{2} = 1 - \frac{1}{2^{n_0+1}} > 1 - \eta(\varepsilon, T^*).$$

By hypothesis, there is $(\widetilde{y}, \tilde{z}) \in \Pi(\ell_1)$ such that $|\langle \tilde{z}, T^*\widetilde{y}\rangle| = 1, \|\widetilde{y} - y\|_1 < \varepsilon$ and $\|\tilde{z} - z\|_\infty < \varepsilon$. However, observe that we then have

$$1 = \left| \sum_{j=1}^{\infty} \frac{1}{2^j}\tilde{z}(j)\widetilde{y}(1) + \tilde{z}(2)\widetilde{y}(2) \right| \leqslant \sum_{j=1}^{\infty} \frac{1}{2^j}|\tilde{z}(j)||\widetilde{y}(1)| + |\tilde{z}(2)||\widetilde{y}(2)|$$
$$\leqslant |\widetilde{y}(1)| + |\widetilde{y}(2)| \leqslant 1.$$

This implies that $\sum_{j=1}^{\infty} \frac{1}{2^j}|\tilde{z}(j)||\widetilde{y}(1)| = |\widetilde{y}(1)|$ and $|\tilde{z}(2)||\widetilde{y}(2)| = |\widetilde{y}(2)|$. If $\widetilde{y}(1) = 0$, then $\|\widetilde{y} - y\| \geqslant 1/2$ which is a contradiction. So, $\widetilde{y}(1) \neq 0$. Then $|\tilde{z}(j)| = 1$ for all $j \in \mathbb{N}$. This implies that $\|\tilde{z} - z\|_\infty \geqslant 1$, a contradiction again. Hence, $T^* \notin \mathcal{A}_{\rm nu}(X^*)$.

It is natural to ask whether the canonical projections P_N on a Banach space X with Schauder basis are elements of the set $\mathcal{A}_{\|\cdot\|}(X,X)$ or $\mathcal{A}_{\rm nu}(X)$. The following result answers this question positively when X is a classical sequence space c_0 or ℓ_p with $1 \leqslant p < \infty$.

Theorem 2.14. *Let $X = c_0$ or ℓ_p with $1 \leqslant p < \infty$. Then, $P_N \in \mathcal{A}_{\|\cdot\|}(X,X)$ and $P_N \in \mathcal{A}_{nu}(X)$.*

Proof. Let us notice first that if $1 < p < \infty$, we are in the assumptions of Theorem 2.5. So, $P_N \in \mathcal{A}_{\|\cdot\|}(\ell_p, \ell_p)$ and $P_N \in \mathcal{A}_{\rm nu}(\ell_p)$ when $1 < p < \infty$. Now, let us see that $P_N \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$. Indeed, it is clear that $\|P_N\| = 1$ and it always attains the norm. Given $\varepsilon > 0$, if $x \in S_{\ell_1}$ satisfies

$$1 - \sum_{j\geqslant N+1} |x(j)| = \sum_{j=1}^{N} |x(j)| = \|P_N(x)\|_1 > 1 - \varepsilon,$$

then $\sum_{j\geqslant N+1}|x(j)|<\varepsilon$. Now, we define

$$\overline{x} := \left(\sum_{j=1}^{N}|x(j)|\right)^{-1}(x(1),\ldots,x(N),0,0,\ldots)\in S_{\ell_1}.$$

Then, $\|\overline{x}\|_1 = 1 = \|P_N(\overline{x})\|_1$ and $\overline{x}$ is close to x in the ℓ_1-norm. To see that $P_N \in \mathcal{A}_{\|\cdot\|}(c_0,c_0)$, note that if $x\in S_{c_0}$ satisfies $\|(x(j))_{j=1}^N\|_\infty = \|P_N(x)\|_\infty > 1-\varepsilon$, then there is $j_0\in\mathbb{N}$ with $j_0\leqslant N$ such that $|x(j_0)|>1-\varepsilon$. Then, we may define

$$\overline{x} := \left(x(1),\ldots,x(j_0-1),\frac{x(j_0)}{|x(j_0)|},x(j_0+1),\ldots,x(N),\ldots\right)\in S_{c_0}.$$

Then, $\|\overline{x}-x\|_\infty = 1-|x(j_0)|<\varepsilon$ and $\|P_N(\overline{x})\|_\infty = 1$.

Now, let us prove by contradiction that $P_N\in\mathcal{A}_{\mathrm{nu}}(c_0)$. If this is not true, there are $\varepsilon_0>0$ and, for each $n\in\mathbb{N}$, $(x_n,x_n^*)\in\Pi(c_0)$ such that

$$1\geqslant|\langle x_n^*,P_N(x_n)\rangle|\geqslant 1-\frac{1}{n} \tag{7}$$

and if $(x,x^*)\in\Pi(c_0)$ satisfies $\|x-x_n\|_\infty<\varepsilon_0$ and $\|x^*-x_n^*\|_1<\varepsilon_0$, then $|\langle x^*,P_N(x)\rangle|<1$. By (7), we have that

$$1\geqslant|x_n^*(1)x_n(1)+\ldots+x_n^*(N)x_n(N)|\geqslant 1-\frac{1}{n} \tag{8}$$

for each $n\in\mathbb{N}$. Now, since $((x_n(1),\ldots,x_n(N)))_{n=1}^\infty\subset B_{\ell_\infty^N}$ and $((x_n^*(1),\ldots,x_n^*(N)))_{n=1}^\infty\subset B_{\ell_1^N}$, by compactness and passing to subsequences if necessary, there are two elements $(x_\infty(1),\ldots,x_\infty(N))\in B_{\ell_\infty^N}$ and $(x_\infty^*(1),\ldots,x_\infty^*(N))\in B_{\ell_1^N}$ such that $(x_n(1),\ldots,x_n(N))\to(x_\infty(1),\ldots,x_\infty(N))$ and $(x_n^*(1),\ldots,x_n^*(N))\to(x_\infty^*(1),\ldots,x_\infty^*(N))$ when $n\to\infty$.

Now, since $(x_n,x_n^*)\in\Pi(c_0)$ for each $n\in\mathbb{N}$, by (7), we get that

$$\left|\sum_{j\geqslant N+1}x_n^*(j)x_n(j)\right|\leqslant\sum_{j\geqslant N+1}|x_n^*(j)| = 1-\sum_{j=1}^N|x_n^*(j)|\leqslant\frac{1}{n}$$

for each $n\in\mathbb{N}$. So,

$$\begin{aligned}\left|1-\sum_{j=1}^N x_\infty^*(j)x_\infty(j)\right| &= \left|\sum_{j=1}^\infty x_n^*(j)x_n(j)-\sum_{j=1}^N x_\infty^*(j)x_\infty(j)\right| \\ &\leqslant \left|\sum_{j=1}^N x_n^*(j)x_n(j)-\sum_{j=1}^N x_\infty^*(j)x_\infty(j)\right|+\left|\sum_{j\geqslant N+1}x_n^*(j)x_n(j)\right| \\ &\leqslant \sum_{j=1}^N|x_n^*(j)-x_\infty^*(j)||x_n(j)|+\sum_{j=1}^N|x_\infty^*(j)||x_n(j)-x_\infty(j)|+\frac{1}{n} \\ &\leqslant \|x_n^*-x_\infty^*\|_1\|x_n\|_\infty+\|x_\infty^*\|_1\|x_n-x_\infty\|_\infty+\frac{1}{n} \\ &\leqslant \|x_n^*-x_\infty^*\|_1+\|x_n-x_\infty\|_\infty+\frac{1}{n}\overset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

So, $\sum_{j=1}^N x_\infty^*(j)x_\infty(j)=1$.

Finally, consider

$$\overline{x}_n := (x_\infty(1),\ldots,x_\infty(N),x_n(N+1),x_n(N+1),\ldots)\in B_{c_0}$$

and

$$\overline{x}^* := (x_\infty^*(1),\ldots,x_\infty^*(N),0,0,\ldots)\in B_{\ell_1}.$$

Then, since $\sum_{j=1}^N x_\infty^*(j)x_\infty(j)=1$, we have that $\langle\overline{x}^*,\overline{x}_n\rangle=1$. In particular, $(\overline{x}_n,\overline{x}^*)\in\Pi(c_0)$. Moreover,

(i) $|\langle \overline{x}^*, P_N(\overline{x}_n)\rangle| = 1$,

(ii) $\|\overline{x}^* - x_n^*\|_1 \leqslant \sum_{i=1}^{N} |x_n^*(i) - x_\infty^*(i)| + \frac{1}{n}$, and

(iii) $\|\overline{x}_n - x_n\|_\infty = \sup_{1\leqslant i \leqslant N} |x_n(i) - x_\infty(i)|$.

Now, to get a contradiction, we consider n large enough to make $\|\overline{x}^* - x_n^*\|_1$ and $\|\overline{x}_n - x_n\|_\infty$ small. So, $P_N \in \mathcal{A}_{\text{nu}}(c_0)$ as desired. The proof that $P_N \in \mathcal{A}_{\text{nu}}(\ell_1)$ is very similar to the last one and we omit it. □

3. Connecting the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{nu}$

In this section, we introduce a natural approach to connect the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$ through direct sums. If we have an operator $T \in \mathcal{L}(W, Z)$, then there is the simplest way to define $\widetilde{T} : W \oplus Z \to W \oplus Z$: consider $\widetilde{T} := \iota_2 \circ T \circ P_1$, that is, $\widetilde{T}(w, z) = (0, Tw)$ for every $(w, z) \in W \oplus Z$. Conversely, if we have an operator $S \in \mathcal{L}(W \oplus Z)$, then we can consider $\breve{S} : W \to Z$ defined as $\breve{S} := P_2 \circ S \circ \iota_1$, that is, $\breve{S}(w) = (P_2 \circ S)(w, 0)$ for every $w \in W$. We start with the following result that shows that under some assumptions one can obtain an operator S in $\mathcal{A}_{\text{nu}}(W \oplus Z)$ whenever T belongs to $\mathcal{A}_{\|\cdot\|}(W, Z)$ where S is generated by T, i.e., $S = \widetilde{T}$.

Theorem 3.1. *Let W, Z be uniformly smooth Banach spaces. If $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$, then $\widetilde{T} \in \mathcal{A}_{\text{nu}}(W \oplus_1 Z)$.*

Proof. Suppose $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$. It is plain to check that $\widetilde{T}$ attains its numerical radius and $\nu(\widetilde{T}) = 1$. Given $\varepsilon \in (0, 1)$, we set

$$\eta(\varepsilon, \widetilde{T}) := \min\left\{\eta\left(\min\left\{\frac{\delta_{W^*}(\varepsilon)}{2}, \frac{\delta_{Z^*}(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}, T\right), \frac{\delta_{W^*}(\varepsilon)}{2}, \frac{\delta_{Z^*}(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\} > 0,$$

where $\varepsilon \mapsto \delta_{W^*}(\varepsilon)$ and $\varepsilon \mapsto \delta_{Z^*}(\varepsilon)$ are the modulus of convexity of W^* and Z^*, respectively. Let $((w_1, z_1), (w_1^*, z_1^*)) \in \Pi(W \oplus_1 Z)$ be such that

$$|\langle z_1^*, T(w_1)\rangle| = \left|\langle (w_1^*, z_1^*), \widetilde{T}(w_1, z_1)\rangle\right| > 1 - \eta(\varepsilon, \widetilde{T}).$$

Since $w_1 \in B_W$ and

$$\|T(w_1)\| \geqslant |\langle z_1^*, T(w_1)\rangle| > 1 - \eta\left(\min\left\{\frac{\delta_{W^*}(\varepsilon)}{2}, \frac{\delta_{Z^*}(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}, T\right),$$

there is $w_2 \in S_W$ such that

$$\|T(w_2)\| = 1 \quad \text{and} \quad \|w_2 - w_1\| < \min\left\{\frac{\delta_{W^*}(\varepsilon)}{2}, \frac{\delta_{Z^*}(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}. \tag{9}$$

Now let us notice that since $\|w_1\| + \|z_1\| = 1$ and $\|w_1\| \geqslant |\langle z_1^*, T(w_1)\rangle| > 1 - \eta(\varepsilon, \widetilde{T})$, we have that

$$\|z_1\| = 1 - \|w_1\| < \eta(\varepsilon, \widetilde{T}). \tag{10}$$

Let $w_2^* \in S_{W^*}$ be such that $\langle w_2^*, w_2\rangle = 1$. Since $\|z_1\| < \eta(\varepsilon, \widetilde{T})$, we have that $|\langle z_1^*, z_1\rangle| \leqslant \|z_1^*\|\|z_1\| < \eta(\varepsilon, \widetilde{T})$. Then,

$$\left|\frac{\langle z_1^*, z_1\rangle - \langle w_1^*, w_2 - w_1\rangle}{2}\right| \leqslant |\langle z_1^*, z_1\rangle| + \|w_1^*\|\|w_2 - w_1\| < \delta_{W^*}(\varepsilon).$$

So, we have

$$\begin{aligned}\left\|\frac{w_1^*+w_2^*}{2}\right\| \geqslant \left|\left\langle\frac{w_1^*+w_2^*}{2}, w_2\right\rangle\right| &= \left|\frac{1+\langle w_1^*, w_1\rangle+\langle w_1^*, w_2-w_1\rangle}{2}\right| \\ &= \left|\frac{2-\langle z_1^*, z_1\rangle+\langle w_1^*, w_2-w_1\rangle}{2}\right| \\ &\geqslant 1-\left|\left(\frac{\langle z_1^*, z_1\rangle-\langle w_1^*, w_2-w_1\rangle}{2}\right)\right| \\ &> 1-\delta_{W^*}(\varepsilon)\end{aligned}$$

and taking into account that W^* is uniformly convex, we have that

$$\|w_2^*-w_1^*\|<\varepsilon. \tag{11}$$

Let $\theta\in\mathbb{R}$ be such that $\langle z_1^*, T(w_2)\rangle = e^{i\theta}|\langle z_1^*, T(w_2)\rangle|$. Notice that

$$\begin{aligned}|\langle z_1^*, T(w_2)\rangle| = |\langle z_1^*, T(w_1)\rangle+\langle z_1^*, T(w_2-w_1)\rangle| &\geqslant |\langle z_1^*, T(w_1)\rangle|-|\langle z_1^*, T(w_2-w_1)\rangle| \\ &\geqslant 1-\delta_{Z^*}(\varepsilon).\end{aligned}$$

Now, let $z_2^*\in S_{Z^*}$ be such that

$$\langle z_2^*, T(w_2)\rangle = e^{i\theta}. \tag{12}$$

Observe that

$$\left\|\frac{z_1^*+z_2^*}{2}\right\| \geqslant \left|\left\langle\frac{z_1^*+z_2^*}{2}, T(w_2)\right\rangle\right| = \left|\frac{e^{i\theta}|\langle z_1^*, T(w_2)\rangle|+e^{i\theta}}{2}\right| = \frac{1+|\langle z_1^*, T(w_2)\rangle|}{2} > 1-\delta_{Z^*}(\varepsilon).$$

which implies that

$$\|z_2^*-z_1^*\|<\varepsilon. \tag{13}$$

Finally, considering the point $((w_2,0),(w_2^*,z_2^*))\in\Pi(W\oplus_1 Z)$, we conclude that $\widetilde{T}\in\mathcal{A}_{\text{nu}}(W\oplus_1 Z)$ in view of:

(i) $\|(w_2,0)-(w_1,z_1)\|_1 = \|w_2-w_1\|+\|z_1\| \overset{(9),(10)}{<} \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$

(ii) $\|(w_2^*,z_2^*)-(w_1^*,z_1^*)\|_\infty = \max\{\|w_2^*-w_1^*\|, \|z_2^*-z_1^*\|\} \overset{(11),(13)}{<} \varepsilon.$

(iii) $\left|\langle(w_2^*,z_2^*), \widetilde{T}(w_2,0)\rangle\right| = |\langle z_2^*, T(w_2)\rangle| \overset{(12)}{=} 1.$

∎

Remark 3.2. Theorem 3.1 is no longer true if we consider general Banach spaces instead of uniformly smooth ones. Indeed, consider the real Banach space ℓ_1. Example 2.8 provides an operator that belongs to $\mathcal{A}_{\|\cdot\|}(\ell_1,\ell_1)$ but not to $\mathcal{A}_{\text{nu}}(\ell_1)$. We will show that this operator does not satisfy the property stated in Theorem 3.1. Indeed, let $T:\ell_1\longrightarrow\ell_1$ be defined as

$$T(x) := \sum_{j=1}^{\infty}\frac{x(1)}{2^j}e_j, \qquad \text{for all } x\in\ell_1,$$

and consider $\widetilde{T}\in\mathcal{L}(\ell_1\oplus_1\ell_1)$. Note that if $((x,y),(x^*,y^*))\in\Pi(\ell_1\oplus_1\ell_1)$ satisfies

$$|\langle(x^*,y^*), \widetilde{T}(x,y)\rangle| = |\langle y^*, T(x)\rangle| = \left|\sum_{j=1}^{\infty}\frac{y^*(j)x(1)}{2^j}\right| = 1, \tag{14}$$

then, since $|y^*(j)x(1)|\leqslant 1$ for all $j\in\mathbb{N}$, one gets easily that $y^*(j)x(1)$ has to be equal to either 1 or -1 for all $j\in\mathbb{N}$. From here and the fact that $((x,y),(x^*,y^*))\in\Pi(\ell_1\oplus_1\ell_1)$, we get that the only possibilities have the form $x=se_1$, $y=0$, $x^*=(s,x^*(2),x^*(3),\ldots)$, and $y^*=(r,r,r,\ldots)$ with $|x^*(j)|\leqslant 1$ for all $j>1$, where $s,r\in\{-1,1\}$.

Now, suppose by contradiction that for a given $\varepsilon \in (0,1)$, there is $\eta(\varepsilon, \widetilde{T}) > 0$. Let $n_0 \in \mathbb{N}$ be such that

$$\sum_{j=1}^{n_0} \frac{1}{2^j} > 1 - \eta(\varepsilon, \widetilde{T}),$$

and set $w = e_1$, $z = 0$, $w^* = e_1^*$, and $z^* = e_1^* + \ldots + e_{n_0}^*$. It is immediate to check that $((w,z),(w^*,z^*)) \in \Pi(\ell_1 \oplus_1 \ell_1)$ and also that $|\langle (w^*,z^*), \widetilde{T}(w,z)\rangle| > 1 - \eta(\varepsilon, \widetilde{T})$. Then, there must be some $((x,y),(x^*,y^*)) \in \Pi(\ell_1 \oplus_1 \ell_1)$ satisfying (14) and such that $\|(w,z)-(x,y)\|_1 < \varepsilon$ and $\|(w^*,z^*)-(x^*,y^*)\|_\infty < \varepsilon$. But this is already a contradiction, since

$$\varepsilon > \|(x^*-w^*, y^*-z^*)\|_\infty \geqslant \|y^*-z^*\|_\infty \geqslant |y^*(n_0+1) - z^*(n_0+1)| \geqslant 1.$$

Therefore $\widetilde{T} \notin \mathcal{A}_{\mathrm{nu}}(\ell_1 \oplus_1 \ell_1)$ as desired, even though $T \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$.

We recall that an absolute norm $|\cdot|_a$ is a norm in $\mathbb{R}^2$ which satisfies $|(1,0)|_a = |(0,1)|_a = 1$ and $|(u,v)|_a = |(|u|,|v|)|_a$ for every $u, v \in \mathbb{R}$. The absolute sum of two Banach spaces W and Z with respect to $|\cdot|_a$ is the Banach space $W \times Z$ endowed with the norm $\|(w,z)\|_a = |(\|w\|, \|z\|)|_a$ for every $w \in W$ and every $z \in Z$. We denote it as $W \oplus_a Z$. We say that $|\cdot|_a$ is of type 1 if the vector $(1,0)$ is a vertex of $B_{(\mathbb{R}^2, \|\cdot\|_a)}$. Equivalently, $|\cdot|_a$ is of type 1 if and only there is a positive number $K > 0$ such that $|u| + K|v| \leqslant |(u,v)|_a$ for every $u, v \in \mathbb{R}$ (see [14, Propositions 5.5 and 5.6]). Note that the ℓ_1-norm is of type 1.

Remark 3.3. It is natural to ask if we can replace ℓ_1-sum in Theorem 3.1 by an absolute sum of type 1: if $T \in \mathcal{A}_{\|\cdot\|}(W,Z)$, then it is true that $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_a Z)$, where $|\cdot|_a$ is of type 1? We show that such a generalization cannot be done by presenting a concrete example. Consider the norm $|\!|\!|\cdot|\!|\!|$ on $\mathbb{R}^2$ defined as $|\!|\!|(p,q)|\!|\!| = |p| + (1/2)|q|$ for every $(p,q) \in \mathbb{R}^2$. Then the norm $|\!|\!|\cdot|\!|\!|$ is of type 1 (with constant $K = 1/2$).

Suppose that T belongs to $\mathcal{A}_{\|\cdot\|}(W,Z)$. To see that $\widetilde{T}$ belongs to $\mathcal{A}_{\mathrm{nu}}(W \oplus_a Z)$, where $\|\cdot\|_a$ is the type 1 absolute norm induced from the norm $|\!|\!|\cdot|\!|\!|$, we have to check that $\widetilde{T}$ has its numerical radius 1 and attains its numerical radius. Notice that $\|(w,z)\|_a = \|w\| + (1/2)\|z\|$ for every $(w,z) \in W \oplus_a Z$ and it is easy to observe that $\|(w^*,z^*)\|_{a*} = \max\{\|w^*\|, 2\|z^*\|\}$ for every $(w^*,z^*) \in W^* \oplus_{a*} Z^*$. Since

$$\begin{aligned} \left|\langle (w^*,z^*), \widetilde{T}(w,z)\rangle\right| = |\langle (w^*,z^*), (0,Tw)\rangle| &= |\langle z^*(Tw)\rangle| \\ &\leqslant \|z^*\| \\ &\leqslant \frac{1}{2}\|(w^*,z^*)\|_{a*} = \frac{1}{2} \end{aligned}$$

for every $((w,z),(w^*,z^*)) \in \Pi(W \oplus_a Z)$, we have that $\nu(\widetilde{T}) \leqslant 1/2$. This shows that $\widetilde{T}$ cannot belong to $\mathcal{A}_{\mathrm{nu}}(W \oplus_a Z)$.

Remark 3.4. The converse of Theorem 3.1 is not true. Indeed, let $S : \ell_2 \oplus_1 \ell_2 \to \ell_2 \oplus_1 \ell_2$ be defined as

$$S(x,y) = ((x(1),0,0,\cdots),(0,0,0,\cdots)), \quad \forall (x,y) \in \ell_2 \oplus_1 \ell_2,$$

where ℓ_2 is a real space. Note that $\nu(S) = 1$ and S attains its numerical radius. For $\varepsilon \in (0,1)$, suppose that $|\langle (x^*,y^*), S(x,y)\rangle| > 1 - \varepsilon > 0$ for some $((x,y),(x^*,y^*)) \in \Pi(\ell_2 \oplus_1 \ell_2)$. Then $|x^*(1)x(1)| > 1-\varepsilon$; hence $|x(1)| > 1-\varepsilon$ and $|x^*(1)| > 1-\varepsilon$. Note that

$$1 = \left(\sum_n |x(n)|^2\right)^{1/2} + \left(\sum_n |y(n)|^2\right)^{1/2} \geqslant |x(1)| + \left(\sum_n |y(n)|^2\right)^{1/2} \geqslant |x(1)| > 1-\varepsilon$$

which implies that $\left(\sum_n |y(n)|^2\right)^{1/2} < \varepsilon$. Note also that

$$1 \geqslant \sum_n |x(n)|^2 = |x(1)|^2 + \sum_{n\neq 1} |x(n)|^2 \geqslant |x(1)|^2 > (1-\varepsilon)^2$$

which implies that $(\sum_{n\neq 1}|x(n)|^2)^{1/2} < (2\varepsilon-\varepsilon^2)^{1/2}$.

On the other hand,

$$\begin{aligned} 1=\sum_n x(n)x^*(n)+\sum_n y(n)y^*(n) &\leqslant \left(\sum_n |x(n)|^2\right)^{1/2}\left(\sum_n |x(^*n)|^2\right)^{1/2}+\\ &\quad+\left(\sum_n |y(n)|^2\right)^{1/2}\left(\sum_n |y^*(n)|^2\right)^{1/2} \\ &\leqslant \left(\sum_n |x(n)|^2\right)^{1/2}+\left(\sum_n |y(n)|^2\right)^{1/2} \\ &= 1 \end{aligned}$$

From this, we have

$$\left(\sum_n |x^*(n)|^2\right)^{1/2}=\left(\sum_n |y^*(n)|^2\right)^{1/2}=1.$$

As above, we can conclude that $(\sum_{n\neq 1}|x^*(n)|^2)^{1/2} < (2\varepsilon-\varepsilon^2)^{1/2}$. If we define pairs of vectors

$$\begin{aligned} (\widetilde{x},\widetilde{y}) &= \left(\left(\frac{x(1)}{|x(1)|},0,0,\cdots\right),(0,0,0,\cdots)\right)\in S_{\ell_2\oplus_1\ell_2} \\ (\widetilde{x^*},\widetilde{y^*}) &= \left(\left(\frac{x^*(1)}{|x^*(1)|},0,0,\cdots\right),(y^*(1),y^*(2),y^*(3),\cdots)\right)\in S_{\ell_2\oplus_\infty\ell_2}, \end{aligned}$$

then

$$\|(x,y)-(\widetilde{x},\widetilde{y})\|=\left((1-|x(1)|)^2+\sum_{n\neq 1}|x(n)|^2\right)^{1/2}+\left(\sum_n |y(n)|^2\right)^{1/2}\leqslant \varepsilon+\sqrt{2\varepsilon}$$

$$\|(x^*,y^*)-(\widetilde{x^*},\widetilde{y^*})\|=\max\left\{\left((1-|x^*(1)|)^2+\sum_{n\neq 1}|x^*(n)|^2\right)^{1/2},0\right\}\leqslant\sqrt{2\varepsilon}.$$

It is clear that $\left((\widetilde{x},\widetilde{y}),(\widetilde{x^*},\widetilde{y^*})\right)\in\Pi(\ell_2\oplus_1\ell_2)$ and that $|\langle(\widetilde{x^*},\widetilde{y^*}),S(\widetilde{x},\widetilde{y})\rangle|=1$. Thus, S belongs to $\mathcal{A}_{\mathrm{nu}}(\ell_2\oplus_1\ell_2)$. However, $\check{S}:\ell_2\to\ell_2$ is the operator such that

$$\check{S}x=(P_2\circ S)(x,0)=P_2((x(1),0,0,\cdots),(0,0,0,\cdots))=0$$

for every $x\in\ell_2$; hence $\check{S}=0$ and the null operator cannot belong to $\mathcal{A}_{\|\cdot\|}(\ell_2,\ell_2)$.

Next we prove the analogous result for ℓ_∞-sum but under different hypothesis on the underlying spaces.

Theorem 3.5. *Let Z be a uniformly convex and uniformly smooth Banach space. Let W be an arbitrary Banach space. If $T\in\mathcal{A}_{\|\cdot\|}(W,Z)$, then $\widetilde{T}\in\mathcal{A}_{\mathrm{nu}}(W\oplus_\infty Z)$.*

Proof. Suppose $T\in\mathcal{A}_{\|\cdot\|}(W,Z)$. It is not difficult to see that $\nu(\widetilde{T})=1$ and that $\widetilde{T}$ attains its numerical radius. Now let $\varepsilon\in(0,1)$ be given and set $\eta(\varepsilon,\widetilde{T})$ as the positive real number

$$\eta(\varepsilon,\widetilde{T}):=\min\{\varepsilon_0,\eta(\varepsilon_0,T)\},$$

where

$$\varepsilon_0=\min\left\{\frac{1}{2}\delta_{Z^*}\left(\min\left\{\frac{\delta_Z(\varepsilon)}{2},\frac{\varepsilon}{2}\right\}\right),\frac{\delta_Z(\varepsilon)}{2},\frac{\varepsilon}{2}\right\}.$$

Let $((w_1, z_1), (w_1^*, z_1^*)) \in \Pi(W \oplus_\infty Z)$ be such that

$$|\langle z_1^*, T(w_1)\rangle| = \left|\langle (w_1^*, z_1^*), \widetilde{T}(w_1, z_1)\rangle\right| > 1 - \eta(\varepsilon, \widetilde{T}).$$

Since $\|T(w_1)\| \geqslant |\langle z_1^*, T(w_1)\rangle| > 1 - \eta(\varepsilon, \widetilde{T})$, there is $w_2 \in S_W$ such that

$$\|T(w_2)\| = 1 \quad \text{and} \quad \|w_2 - w_1\| < \varepsilon_0. \tag{15}$$

Since $\|w_1^*\| + \|z_1^*\| = 1$ and $\|z_1^*\| \geqslant |\langle z_1^*, T(w_1)\rangle| > 1 - \eta(\varepsilon, \widetilde{T})$, we get that

$$\|w_1^*\| = 1 - \|z_1^*\| < \eta(\varepsilon, \widetilde{T}) \leqslant \frac{\varepsilon}{2} \tag{16}$$

Let $\theta \in \mathbb{R}$ be such that $\langle z_1^*, T(w_2)\rangle = |\langle z_1^*, T(w_2)\rangle| e^{i\theta}$. Pick $z_2^* \in S_{Z^*}$ to be such that

$$\langle z_2^*, T(w_2)\rangle = e^{i\theta} \tag{17}$$

and notice that

$$\begin{aligned} |\langle z_1^*, T(w_2)\rangle| &= |\langle z_1^*, T(w_1)\rangle + \langle z_1^*, T(w_2 - w_1)\rangle| \\ &\geqslant |\langle z_1^*, T(w_1)\rangle| - \|w_2 - w_1\| \\ &> 1 - 2\varepsilon_0, \end{aligned} \tag{18}$$

in particular,

$$|\langle z_1^*, T(w_2)\rangle| > 1 - \delta_{Z^*}\left(\min\left\{\frac{\delta_Z(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}\right).$$

Thus,

$$\begin{aligned} \left\|\frac{z_1^* + z_2^*}{2}\right\| \geqslant \left|\left\langle \frac{z_1^* + z_2^*}{2}, T(w_2)\right\rangle\right| &= \left|\frac{|\langle z_1^*, T(w_2)\rangle| e^{i\theta} + e^{i\theta}}{2}\right| \\ &= \frac{|\langle z_1^*, T(w_2)\rangle| + 1}{2} \\ &> 1 - \delta_{Z^*}\left(\min\left\{\frac{\delta_Z(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}\right). \end{aligned} \tag{19}$$

This implies that

$$\|z_2^* - z_1^*\| < \min\left\{\frac{\delta_Z(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\} \tag{20}$$

By (18) and (19),

$$\begin{aligned} \left\|\frac{T(e^{-i\theta} w_2) + z_1}{2}\right\| \geqslant \left|\left\langle z_1^*, \frac{T(e^{-i\theta} w_2) + z_1}{2}\right\rangle\right| &= \left|\frac{|\langle z_1^*, T(w_2)\rangle| + \langle z_1^*, z_1\rangle}{2}\right| \\ &= \left|\frac{|\langle z_1^*, T(w_2)\rangle| + 1 - \langle w_1^*, w_1\rangle}{2}\right| \\ &\geqslant \left|\frac{|\langle z_1^*, T(w_2)\rangle| + 1}{2}\right| - \left|\frac{\langle w_1^*, w_1\rangle}{2}\right| \\ &\geqslant 1 - \delta_Z(\varepsilon) \end{aligned}$$

and so

$$\|T(e^{-i\theta} w_2) - z_1\| < \varepsilon. \tag{21}$$

Finally, consider the point $\big((w_2, T(e^{-i\theta} w_2)), (0, z_2^*)\big) \in \Pi(W \oplus_\infty Z)$. We conclude that $\widetilde{T} \in \mathcal{A}_{\text{nu}}(W \oplus_\infty Z)$ due to the following observations:

(i) $\|(w_2, T(e^{-i\theta} w_2)) - (w_1, z_1)\|_\infty = \max\{\|w_2 - w_1\|, \|T(e^{-i\theta} w_2) - z_1\|\} \overset{(15),(21)}{<} \varepsilon.$

(ii) $\|(0,z_2^*)-(w_1^*,z_1^*)\|_1 = \|w_1^*\| + \|z_2^*-z_1^*\| \overset{(16),(20)}{<} \varepsilon.$

(iii) $\left|\langle (0,z_2^*), \widetilde{T}(w_2, T(e^{-i\theta}w_2))\rangle\right| = |\langle z_2^*, T(w_2)\rangle| \overset{(17)}{=} 1.$

□

Remark 3.6. Similar to what happened on Theorem 3.1, Theorem 3.5 is not true in general. Indeed, consider the real Banach space ℓ_1. Like we did in Remark 3.2, we will show that the operator introduced in Example 2.8 does not satisfy the property stated in Theorem 3.5. Let $T:\ell_1 \longrightarrow \ell_1$ be defined as

$$T(x) := \sum_{j=1}^{\infty} \frac{x(1)}{2^j} e_j, \qquad \text{for all } x \in \ell_1,$$

and let $\widetilde{T} \in \mathcal{L}(\ell_1 \oplus_\infty \ell_1)$ be defined accordingly. First note that if $((x,y),(x^*,y^*)) \in \Pi(\ell_1 \oplus_\infty \ell_1)$, we have:

(i) $\|(x,y)\| = \max\{\|x\|_1, \|y\|_1\} = 1.$
(ii) $\|(x^*,y^*)\| = \|x^*\|_\infty + \|y^*\|_\infty = 1.$
(iii) $\langle x^*, x\rangle + \langle y^*, y\rangle = 1.$

Also, if $((x,y),(x^*,y^*)) \in \Pi(\ell_1 \oplus_\infty \ell_1)$ satisfies

$$|\langle (x^*,y^*), \widetilde{T}(x,y)\rangle| = |\langle y^*, T(x)\rangle| = \left| \sum_{j=1}^{\infty} \frac{y^*(j)x(1)}{2^j} \right| = 1, \tag{22}$$

then, $y^*(j)x(1)$ has to be equal to either 1 or -1 for all $j \in \mathbb{N}$. From here, and using (i), (ii) and (iii), we get that the only possibilities have the form $x = se_1$, $y = (y(1), y(2), y(3), \ldots)$ with $\sum_{j=1}^{\infty} y(j) = r$, x^*0, and $y^* = (r,r,r,\ldots)$, where $s, r \in \{-1, 1\}$.

Suppose by contradiction that for a given $\varepsilon \in (0,1)$, there is $\eta(\varepsilon, \widetilde{T}) > 0$. Let $n_0 \in \mathbb{N}$ be such that

$$\sum_{j=1}^{n_0} \frac{1}{2^j} > 1 - \eta(\varepsilon, \widetilde{T}),$$

and set $w = e_1$, $z = e_1$, $w^* = 0$, and $z^* = e_1^* + \ldots + e_{n_0}^*$. It is immediate to check that $((w,z),(w^*,z^*)) \in \Pi(\ell_1 \oplus_\infty \ell_1)$ and also that $|\langle (w^*,z^*), \widetilde{T}(w,z)\rangle| > 1 - \eta(\varepsilon, \widetilde{T})$. Then, there must be some $((x,y),(x^*,y^*)) \in \Pi(\ell_1 \oplus_\infty \ell_1)$ satisfying (22) and such that $\|(w,z)-(x,y)\|_\infty < \varepsilon$ and $\|(w^*,z^*)-(x^*,y^*)\|_1 < \varepsilon$, but this is already a contradiction, since

$$\varepsilon > \|(x^*-w^*, y^*-z^*)\|_1 \geqslant \|y^*-z^*\|_1 \geqslant |y^*(n_0+1) - z^*(n_0+1)| \geqslant 1.$$

Therefore $\widetilde{T} \notin \mathcal{A}_{\text{nu}}(\ell_1 \oplus_\infty \ell_1)$ as desired, even though $T \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$.

We say that an absolute norm $|\cdot|_a$ in $\mathbb{R}^2$ is of type ∞ if the vector $(1,0)$ is not an extreme point of $B_{(\mathbb{R}^2, \|\cdot\|_a)}$. Equivalently, $|\cdot|_a$ is of type ∞ if and only there exists $b_0 > 0$ such that $|(1,b_0)|_a = 1$.

Remark 3.7. Similar to Remark 3.3, it is not possible to consider an absolute sum of type ∞ instead of ℓ_∞-sum in Theorem 3.5. Indeed, consider the norm $|||\cdot|||$ on $\mathbb{R}^2$ defined as $|||(p,q)||| = \max\{|p|, (1/2)|q|\}$ for every $(p,q) \in \mathbb{R}^2$. Then the norm $|||\cdot|||$ is of type ∞ (with constant $b_0 = 2$).

Suppose that T belongs to $\mathcal{A}_{\|\cdot\|}(W,Z)$. We claim that $\widetilde{T}$ does not belong to $\mathcal{A}_{\text{nu}}(W \oplus_a Z)$, where $\|\cdot\|_a$ is the type ∞ absolute norm induced from the norm $|||\cdot|||$. Note that $\|(w,z)\|_a = \max\{\|w\|, (1/2)\|z\|\}$ for every $(w,z) \in W \oplus_a Z$ and it is plain to observe that $\|(w^*,z^*)\|_{a*} = \|w^*\| + 2\|z^*\|$ for every $(w^*,z^*) \in W^* \oplus_{a*} Z^*$. As we calculated in Remark 3.3,

$$\left|\langle (w^*,z^*), \widetilde{T}(w,z)\rangle\right| \leqslant \frac{1}{2}\|(w^*,z^*)\|_{a*} = \frac{1}{2}$$

for every $((w,z),(w^*,z^*)) \in \Pi(W \oplus_a Z)$, we have that $\nu(\widetilde{T}) \leqslant 1/2$. This proves our claim.

Remark 3.8. The converse of Theorem 3.5 is not true as well. The same argument used in Remark 3.4 shows that $S : \ell_2 \oplus_\infty \ell_2 \to \ell_2 \oplus_\infty \ell_2$, which is defined as

$$S(x, y) = ((x(1), 0, 0, \cdots), (0, 0, 0, \cdots)), \quad \forall (x, y) \in \ell_2 \oplus_\infty \ell_2,$$

where ℓ_2 is a real space, belongs to $\mathcal{A}_{\text{nu}}(\ell_2 \oplus_\infty \ell_2)$. However, $\check{S} = 0$ cannot belong to $\mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$.

We finish the paper by noting that Theorem 3.1 and 3.5 are no longer true for p-sums with $1 < p < \infty$. Indeed, let X be a uniformly convex and uniformly smooth Banach space and consider the identity operator $\text{Id}_X \in \mathcal{L}(X)$. Clearly, Id_X belongs to $\mathcal{A}_{\|\cdot\|}(X, X)$. On the other hand, $\widetilde{\text{Id}}_X : X \oplus_p X \to X \oplus_p X$ is defined as $\widetilde{\text{Id}}_X(x_1, x_2) = (0, x_1)$ for all $x_1, x_2 \in X$. Then $\nu(\widetilde{\text{Id}}_X) \leqslant \|\widetilde{\text{Id}}_X\| = \|\text{Id}_X\| = 1$. If $|\langle (x_1^*, x_2^*), \widetilde{\text{Id}}_X(x_1, x_2)\rangle| = 1$ for some $((x_1, x_2), (x_1^*, x_2^*)) \in \Pi(X \oplus_p X)$, we would have $|\langle x_2^*, x_1\rangle| = 1$, which would imply $\|x_2^*\| = \|x_1\| = 1$. Because of this, we would have $x_1^* = x_2 = 0$ since $\|x_1^*\|^q + \|x_2^*\|^q = 1 = \|x_1\|^p + \|x_2\|^p$ with $\frac{1}{p} + \frac{1}{q} = 1$, contradicting the assumption $\langle x_1^*, x_1\rangle + \langle x_2^*, x_2\rangle = 1$. So, $\widetilde{\text{Id}}_X$ cannot attains its numerical radius; hence cannot belong to $\mathcal{A}_{\text{nu}}(X \oplus_p X)$.

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