SOME KIND OF BISHOP-PHELPS-BOLLOBÁS PROPERTY

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Abstract. In this paper we introduce two Bishop-Phelps-Bollobás type properties for bounded linear operators between two Banach spaces X and Y: property 1 and property 2. These properties are motivated by a Kim-Lee result which states, under our notation, that a Banach space X is uniformly convex if and only if the pair $(X, \mathbb{K})$ satisfies property 2. Positive results of pairs of Banach spaces (X, Y) satisfying property 1 are given and concrete pairs of Banach spaces (X, Y) failing both properties are exhibited. A complete characterization of property 1 for the pairs (ℓ_p, ℓ_q) is also provided.

1. Introduction

Let X and Y be Banach spaces over a real or complex field $\mathbb{K}$. We use the traditional notations S_X and B_X for the unit sphere and the closed unit ball of the space X, respectively. The Banach space of all bounded linear operators $T : X \to Y$ will be represented by $\mathcal{L}(X, Y)$. In particular, when $Y = \mathbb{K}$ we denote $\mathcal{L}(X, \mathbb{K})$ simply by putting X^* which is the dual space of X. We say that an operator $T \in \mathcal{L}(X, Y)$ *attains its norm* if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\| = \sup_{x \in S_X} \|T(x)\|$. In this case, we say that T is *norm attaining* and it attains its norm at x_0. The subset of $\mathcal{L}(X, Y)$ of all norm attaining operators is denoted by $NA(X, Y)$. We recall that a bounded linear operator from X into Y is *compact* if the closure of the image of the unit ball of X is compact in Y. For $1 \leq p \leq \infty$ we denote by $\ell_p^n(\mathbb{K})$ the euclidean space $\mathbb{K}^n$ endowed with the p-norm $\|x\|_p^p := |x_1|^p + \ldots + |x_n|^p$ with $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$ and $1 \leq p < \infty$ and ℓ_∞^n endowed with the sup-norm $\|x\|_\infty := \sup_{j \in \mathbb{N}} |x_j|$. To simplify the notation we put just ℓ_p^n when the field that we are working is specified.

The Bishop-Phelps theorem [4] says that every bounded linear functional can be approximated by norm attaining functionals. In other words, the set of all norm attaining functionals on a Banach space X is dense in its dual space X^*. It was proved by Lindenstrauss [12] in 1963 that, in general, the same result does not work for bounded linear operators. More precisely, he exhibited a Banach space X such that the set $NA(X, X)$ is not dense in $\mathcal{L}(X, X)$. Seven years later, Bollobás proved a numerical version of the Bishop-Phelps theorem which is nowadays known as the Bishop-Phelps-Bollobás theorem [3]. As a consequence of [6, Theorem 2.1] we may enunciate this theorem as follows.

Theorem 1. *(Bishop-Phelps-Bollobás theorem,* [4], [6]*)* Let X be a Banach space. Let $\varepsilon \in (0, 2)$ and suppose that $x_0 \in B_X$ and $x_0^* \in B_{X^*}$ satisfy

$$\operatorname{Re} x_0^*(x_0) > 1 - \frac{\varepsilon^2}{2}.$$

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Then, there are $x_1 \in S_X$ and $x_1^* \in S_{X^*}$ such that

$$|x_1^*(x_1)| = 1, \quad \|x_1 - x_0\| < \varepsilon \ \text{ and } \ \|x_1^* - x_0^*\| < \varepsilon.$$

Since we do not have a Bishop-Phelps version for bounded linear operators, we can not expect a Bishop-Phelps-Bollobás version for this type of functions either. So it is natural to study the conditions that the Banach spaces X and Y must satisfy to get a theorem of this nature. In 2008, Acosta, Aron, García and Maestre introduced a definition in order to attain this problem.

Definition 1. (Bishop-Phelps-Bollobás property, [1]) We say that a pair of Banach spaces (X, Y) satisfies the *Bishop-Phelps-Bollobás property* (BPBp, for short) when given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there are $S \in S_{\mathcal{L}(X,Y)}$ and $x_1 \in S_X$ such that

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

There are many classical Banach spaces that satisfy the BPBp. For example, when X and Y are finite dimensional spaces, the pair (X, Y) has this property [1, Proposition 2.4]. Also, if Y has the property β of Lindenstrauss, as c_0 and ℓ_∞ do, then the pair (X, Y) satisfies the BPBp for all Banach space X [1, Theorem 2.2]. More positive results appear when we assume that the range space Y is uniformly convex: the pairs (ℓ_∞^n, Y), (c_0, Y) and $(L_\infty(\mu), Y)$ all satisfy the BPBp (see [1], [9] and [10], respectively). Also if X is a uniformly convex Banach space, then the pair (X, Y) has the BPBp for all Banach space Y [11, Theorem 3.1].

Just to help make the paper entirely accessible, we remember the concept of uniform convexity. A Banach space X is *uniformly convex* if given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that whenever $x_1, x_2 \in S_X$ satisfy $\|x_1 - x_2\| \geq \varepsilon$, then $\|\frac{1}{2}(x_1 + x_2)\| \leq 1 - \delta(\varepsilon)$. We recall that if $p \in (1, \infty)$ then ℓ_p is uniformly convex (see, for example, [7, Theorem 9.3]).

In 2014, Kim and Lee [11, Theorem 2.1] gave a characterization for uniformly convex Banach spaces that associate this type of spaces with a peculiarity on the Bishop-Phelps-Bollobás property. More precisely, they proved that a Banach space X is uniformly convex if and only if given $\varepsilon > 0$, we are able to find a positive real number $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x_0 \in B_X$ satisfy the relation

$$|x^*(x_0)| > 1 - \eta(\varepsilon),$$

there exists a vector $x_1 \in S_X$ such that

$$|x^*(x_1)| = 1 \quad \text{and} \quad \|x_0 - x_1\| < \varepsilon.$$

Note that the theorem says that a Banach space X is uniformly convex if and only if the pair $(X, \mathbb{K})$ satisfies the Bishop-Phelps-Bollobás property without changing the initial functional x^*, that is, the functional that almost attains its norm at some point x_0 is the same functional that attains its norm at the new vector that is close to x_0.

In this paper, we study this last result for bounded linear operators by considering two properties: property 1 and property 2. First, we study property 1 which consider the positive real number $\eta(\ . \ )$ depending on $\varepsilon > 0$ and also on a fixed operator T, and we get some positives results about it. After that, we study the property in the uniform case and we call it as property 2, that is, when the number η depends only on $\varepsilon > 0$ as we are used to work when we are working with the BPBp. As we will see in the next section, we get some negative results about the uniform case and also we provide uniformly convex Banach spaces X and infinite-dimensional Banach spaces Y such that the pair (X, Y) fails property 1. Finally, we give a complete characterization to property 1 for the pair (ℓ_p, ℓ_q) which describes when these pairs satisfy this property.

2. The results

In this section we study two Bishop-Phelps-Bollobás type properties. Namely, we study the conditions that the Banach spaces X and Y must have and the hypothesis that we have to add to get a Kim-Lee type theorem for bounded linear operators. The Kim-Lee theorem is enunciated as follows.

Theorem 2. *(Kim-Lee theorem, [11])* A Banach space X is uniformly convex if and only if given $\varepsilon > 0$, there exists a positive real number $\eta(\varepsilon) > 0$ such that whenever $x_0^* \in S_{X^*}$ and $x_0 \in B_X$ satisfy

$$|x_0^*(x_0)| > 1 - \eta(\varepsilon),$$

there is $x_1 \in S_X$ such that

$$|x_0^*(x_1)| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

As we mentioned before, it is like a Bishop-Phelps-Bollobás property without changing the initial functional x_0^*. A natural question arises: is the result true for pairs of Banach spaces (X, Y) with X and Y having additional hypothesis considering bounded linear operators instead of bounded linear functionals? Although it is more natural put the question just like that, it is seems to us to be a strong problem in the sense that it will be hard to find concrete pairs of Banach spaces (X, Y) satisfying the Kim-Lee theorem for bounded linear operators. So we start by considering the positive real number $\eta(\ .\ ) > 0$ that appears in Theorem 2 not depending only on $\varepsilon > 0$ but also on a fixed operator T.

Before we do that, we want to comment that Carando, Lassalle and Mazzitelli [5] defined a BPB type property for ideals of multilinear mappings where the positive real number $\eta(\ .\ )$ in the definition of the BPBp depends on a given $\varepsilon > 0$ and also on the ideal norm of the operator defined on a normed ideal of N-linear mappings. In other words, a normed ideal of N-linear mappings $\mathcal{U} = \mathcal{U}(X_1 \times \ldots \times X_N; Y)$ where $X_1, \ldots, X_N, Y$ are Banach spaces has the *weak BPBp* if for each $\Phi \in \mathcal{U}$ with $\|\Phi\| = 1$ and $\varepsilon > 0$, there exists $\eta(\varepsilon, \|\Phi\|_{\mathcal{U}}) > 0$ depending also on $\|\Phi\|_{\mathcal{U}}$ such that if $(x_1, \ldots, x_N) \in S_{X_1} \times \ldots \times S_{X_N}$ satisfies $\|\Phi(x_1, \ldots, x_N)\| > 1 - \eta(\varepsilon, \|\Phi\|_{\mathcal{U}})$, then there exist $\Psi \in \mathcal{U}$ with $\|\Psi\| = 1$ and $(a_1, \ldots, a_N) \in S_{X_1} \times \ldots \times S_{X_N}$ such that

$$\|\Psi(a_1, \ldots, a_N)\| = 1, \quad \|(a_1, \ldots, a_N) - (x_1, \ldots, x_N)\| < \varepsilon \quad \text{and} \quad \|\Psi - \Phi\|_{\mathcal{U}} < \varepsilon.$$

They proved that if $X_1, \ldots, X_N$ are uniformly convex Banach spaces then $\mathcal{U}$ has the weak BPBp for ideals of multilinear mappings for all Banach space Y. Here we will work on a different context where $\eta(\ .\ )$ depends on a fixed operator not on the norm of the operators ideal as we may see in the following definition.

Definition 2. A pair of Banach spaces (X, Y) has *property* 1 if given $\varepsilon > 0$ and $T \in S_{\mathcal{L}(X,Y)}$, there exists $\eta(\varepsilon, T) > 0$ such that whenever $x_0 \in S_X$ satisfies

$$\|T(x_0)\| > 1 - \eta(\varepsilon, T),$$

there is $x_1 \in S_X$ such that

$$\|T(x_1)\| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

If the above property is satisfied for every norm one compact operator, then we say that the pair (X, Y) has *property* 1 *for compact operators*.

Remark 1. Note that in Definition 2 the operator $T : X \longrightarrow Y$ must attains its norm if the pair (X, Y) has property 1. So if X is not reflexive, then the pair (X, Y) fails property 1 for all Banach space Y. Indeed, since X is not reflexive, by James theorem, there is a linear continuous functional $x_0^* \in S_{X^*}$ such that $|x_0^*(x)| < 1$ for all $x \in S_X$. Let $y_0 \in S_Y$ and define $T : X \longrightarrow Y$ by $T(x) := x_0^*(x)y_0$. Then $\|T\| = \|x_0^*\| = 1$ and $\|T(x)\| = |x_0^*(x)| < 1$ for all $x \in S_X$. This implies that T never attains its norm and then the pair (X, Y) can not have property 1.

In the next theorem, we assume that the domain space X is finite dimensional to get the first result about property 1.

Theorem 3. Let X be a finite dimensional Banach space. Then the pair (X, Y) has property 1 for all Banach space Y.

Proof. The proof is by contradiction. Let $T \in S_{\mathcal{L}(X,Y)}$. If the result is false for some $\varepsilon_0 > 0$, then for all $n \in \mathbb{N}$, there exists $x_n \in S_X$ such that

$$1 \geq \|T(x_n)\| > 1 - \frac{1}{n}$$

but $\operatorname{dist}(x_n, NA(T)) \geq \varepsilon_0$ for $n \in \mathbb{N}$, where $NA(T) = \{z \in S_X : \|T(z)\| = 1\}$. Since X is finite dimensional, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \longrightarrow x_0$ for some $x_0 \in X$. This implies that $\|T(x_{n_k})\| \longrightarrow \|T(x_0)\|$ and since

$$1 \geq \|T(x_{n_k})\| \geq 1 - \frac{1}{n}$$

we get that $\|T(x_0)\| = \|x_0\| = 1$ and so $x_0 \in NA(T)$. Then $NA(T) \neq \emptyset$ and

$$\varepsilon \leq \operatorname{dist}(x_{n_k}, NA(T)) \leq \|x_{n_k} - x_0\| \overset{k\to\infty}{\longrightarrow} 0$$

which is a contradiction. So the pair (X, Y) has property 1. □

If we assume that X is uniformly convex, we get that the pair (X, Y) has property 1 for compact operators as we may see in the next theorem.

Theorem 4. Let X be a uniformly convex Banach space. Then the pair (X, Y) has property 1 for compact operators for all Banach space Y.

Proof. The proof is again by contradiction. Let $T \in S_{\mathcal{L}(X,Y)}$ be a compact operator. If the result is false, for some $\varepsilon_0 > 0$ and for all $n \in \mathbb{N}$, there exists $x_n \in S_X$ such that

$$1 \geq \|T(x_n)\| > 1 - \frac{1}{n}$$

but $\operatorname{dist}(x_n, NA(T)) \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Then $\|T(x_n)\| \longrightarrow 1$ as $n \to \infty$. Since X is uniformly convex, X is reflexive and then by the Smulian theorem there exists a subsequence (x_{n_k}) of (x_n) and $x_0 \in X$ such that (x_{n_k}) converges weakly to x_0. Since T is completely continuous (see, for example, [13, Proposition 3.4.34]), $T(x_{n_k})$ converges in norm to $T(x_0)$ as $k \longrightarrow \infty$. Therefore $\|T(x_0)\| = \|x_0\| = 1$ and so $x_0 \in NA(T)$. Thus $NA(T) \neq \emptyset$ and

$$1 \geq \left\| \frac{x_n + x_0}{2} \right\| \geq \left\| \frac{T(x_n) + T(x_0)}{2} \right\| \stackrel{k \to \infty}{\longrightarrow} \|T(x_0)\| = 1.$$

This implies that $\lim_{k \to \infty} \|x_{n_k} + x_0\| = 2$ and, using again that X is uniformly convex, we get that

$$\lim_{k \to \infty} \|x_{n_k} - x_0\| = 0,$$

which is a contradiction because of the following inequalities:

$$\varepsilon_0 \leq \text{dist}(x_{n_k}, NA(T)) \leq \|x_n - x_0\| \stackrel{k \to \infty}{\longrightarrow} 0.$$

□

Consequently we get two more positives results about property 1. In the next corollary we prove that the pair (X, Y) has property 1 whenever X is uniformly convex and Y has the Schur's property which implies as a particular case that the pair (ℓ_2, ℓ_1) satisfies the property. Corollary 2 shows that whenever X is uniformly convex and Y has finite dimension, the pair (X, Y) has property 1 by using the fact that every bounded linear operator with finite dimensional range is compact.

Corollary 1. If X is a uniformly convex Banach space and Y is a Banach space with the Schur's property, then the pair (X, Y) has property 1. In particular, (ℓ_2, ℓ_1) has property 1.

Proof. We apply Theorem 4. To do this, we prove that every bounded linear operator $T : X \longrightarrow Y$ is compact. Indeed, since T is continuous, T is w-w continuous. Let $(x_n)_n \subset B_X$. Since X is reflexive, by the Smulian theorem, there are a subsequence of $(x_n)_n$ (which we denote again by $(x_n)_n$) and $x_0 \in X$ such that $x_n \stackrel{w}{\longrightarrow} x_0$. So $T(x_n) \stackrel{w}{\longrightarrow} T(x_0)$. Now, since Y has the Schur's property, $T(x_n) \longrightarrow T(x_0)$ in norm. So T is compact. By Theorem 4 the pair (X, Y) has property 1. □

Corollary 2. If X is a uniformly convex Banach space and Y is a finite dimensional Banach space, then the pair (X, Y) has property 1.

Note that in the classical definition of the Bishop-Phelps-Bollobás property the number $\eta(\ .\ )$ depends only on $\varepsilon > 0$. So what happen if we ask for more in the definition of property 1? As we will see below, when we put $\eta(\ .\ )$ to depend only on $\varepsilon > 0$ we get negative results. Just to help to make reference we put a name of it.

Definition 3. We say that a pair of Banach spaces (X, Y) has *property* 2 if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there is $x_1 \in S_X$ such that

$$\|T(x_1)\| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

We observe that the Kim-Lee theorem says that a Banach space X is uniformly convex if and only if the pair $(X, \mathbb{K})$ has property 2 where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Note also that if the pair (X, Y) satisfies property 2 then the pair (X, Y) satisfies the BPBp. The first thing that we notice is that if (X, Y) has property 2 for some Banach space Y, then the Banach space X must be uniformly convex.

Proposition 1. Let X be a Banach space. If there exists a Banach space Y such that the pair (X, Y) has property 2, then so does the pair $(X, \mathbb{K})$.

Proof. Given $\varepsilon \in (0,1)$, consider $\eta(\varepsilon) > 0$ the positive real number that satisfies property 2 for the pair (X, Y). We prove that the pair $(X, \mathbb{K})$ has the same property with $\eta(\varepsilon)$. Indeed, let $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ be such that

$$|x_0^*(x_0)| > 1 - \eta(\varepsilon).$$

Let $y_0 \in S_Y$ and define $T \in \mathcal{L}(X, Y)$ by $T(x) := x_0^*(x)y_0$ for all $x \in X$. So $\|T\| = \|x_0^*\| = 1$ and

$$\|T(x_0)\| = |x_0^*(x_0)| > 1 - \eta(\varepsilon).$$

Since the pair (X, Y) has property 2 with $\eta(\varepsilon)$, there exists $x_1 \in S_X$ such that $\|T(x_1)\| = 1$ and $\|x_0 - x_1\| < \varepsilon$. Since $\|T(x_1)\| = |x_0^*(x_1)|$ the proof is complete. $\square$

By the Kim-Lee theorem, we have the following consequence.

Corollary 3. Let X be a Banach space. If there exists a Banach space Y such that the pair (X, Y) has property 2, then X is uniformly convex.

By this last corollary, since ℓ_1^2 is not uniformly convex, all the pairs (ℓ_1^2, Y) fail property 2 for any Banach space Y. What about the converse of Corollary 3? The first thing that come to mind, since every Hilbert space is uniformly convex, is to assume that the domain space X is a Hilbert space and try to find some Banach space Y such that the pair (X, Y) satisfies the property. But even in the simplest situation the result fails as we may see in the next example.

Example 1. This example works for both real and complex cases. For a given $\varepsilon > 0$, suppose that there exists $\eta(\varepsilon) > 0$ satisfying property 2 for the pair $(\ell_2^2, \ell_\infty^2)$. Let $T : \ell_2^2 \to \ell_\infty^2$ be defined by

$$T(x, y) := \left(\left(1 - \frac{1}{2}\eta(\varepsilon) \right) x, y \right)$$

for every $(x, y) \in \ell_2^2$. For every $(x, y) \in B_{\ell_2^2}$, we have

$$\|T(x, y)\|_\infty = \left\| \left(\left(1 - \frac{1}{2}\eta(\varepsilon) \right) x, y \right) \right\|_\infty \leq 1.$$

Since $T(e_2) = 1$, we obtain $\|T\| = 1$. Moreover,

$$\|T(e_1)\| = 1 - \frac{1}{2}\eta(\varepsilon) > 1 - \eta(\varepsilon).$$

We prove now that every $z = (a, b) \in S_{\ell_2^2}$ such that $\|T(z)\|_\infty = 1$ assumes the form $z = \lambda e_2$ for $|\lambda| = 1$. Indeed, since $\left|1 - \frac{1}{2}\eta(\varepsilon)\right| < 1$ and $\|T(z)\|_\infty = 1$, we have $|b| = 1$. Since $|a|^2 + |b|^2 = 1$, we have $a = 0$ and $b = \lambda$ with $|\lambda| = 1$. In summary, we have a norm one operator T and a norm one vector e_1 satisfying $\|T(e_1)\| > 1 - \eta(\varepsilon)$ but if T attains its norm at some point $z \in S_{\ell_2^2}$ then $z = (0, \lambda)$ with $|\lambda| = 1$. This contradicts the assumption that the pair $(\ell_2^2, \ell_\infty^2)$ has property 2 since z is far from e_1 in view of the fact that $\|e_1 - z\|_2 = \|(1, \lambda)\|_2 = \sqrt{2}$.

This shows that the pair $(\ell_2^2(\mathbb{K}), \ell_\infty^2(\mathbb{K}))$ fails to have property 2 for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Now what if we add on the hypothesis that both X and Y are Hilbert spaces? The answer for this question is still no as we can see below.

Proposition 2. Let $1 < p \leq q < \infty$ (or $p < q = \infty$). Given $\beta \in (0,1)$, there exists a bounded linear operator $T_\beta : \ell_p^2 \to \ell_q^2$ with $\|T_\beta\| = 1$ such that

(i) $\|T_\beta(e_1)\|_q = \beta$ and
(ii) for every $z \in S_{\ell_p^2}$ such that $\|T_\beta(z)\|_q = 1$, we have $\|z - e_1\|_p = 2^{\frac{1}{p}}$.

Proof. Let $\beta \in (0,1)$ and $1 < p \leq q < \infty$. Define $T_\beta : \ell_p^2 \to \ell_q^2$ by $T_\beta(x,y) := (\beta x, y)$ for every $(x,y) \in \ell_p^2$. If $\|(x,y)\|_p = 1$, since $p \leq q$, we get

$$\|T_\beta(x,y)\|_q = (\beta^q|x|^q + |y|^q)^{\frac{1}{q}} < (|x|^p + |y|^p)^{\frac{1}{q}} = 1,$$

which implies that $\|T_\beta\| \leq 1$. Since $\|T_\beta(e_2)\|_q = \|e_2\|_q = 1$, we have $\|T_\beta\| = 1$. Now, let $z = (a,b) \in S_{\ell_p^2}$ be such that $\|T_\beta(z)\|_q = 1$. We prove that $b = \lambda e_2$ with $|\lambda| = 1$. Indeed, the equality $\|T_\beta(a,b)\|_q = 1$ implies that $\beta^q|a|^q + |b|^q = 1$ and since $|a|^p + |b|^p = 1$, we do the difference between these two equalities to get

$$(|a|^p - \beta^q|a|^q) + (|b|^p - |b|^q) = 0.$$

Since $p \leq q$ and $|a|, |b| \leq 1$, $|a|^p - \beta^q|a|^q \geq 0$ and $|b|^p - |b|^q \geq 0$. Because of the above equality, we get that $|a|^p - \beta^q|a|^q = 0 = |b|^p - |b|^q$. But $|a|^q \leq |a|^p$ which implies that

$$0 = |a|^p - \beta^q|a|^q \geq (1 - \beta^q)|a|^p.$$

Thus $a = 0$ and then $b = \lambda e_2$ with $|\lambda| = 1$ as desired. So if $z \in S_{\ell_p^2}$ is such that $\|T_\beta(z)\|_q = 1$, then $\|z - e_1\|_p = 2^{\frac{1}{p}}$ which completes the proof. □

As a consequence of this last result, we get that all the pairs $(\ell_p(\mathbb{K}), \ell_q(\mathbb{K}))$ fail property 2 for $1 < p \leq q < \infty$ when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In particular, the pair (ℓ_2^2, ℓ_2^2) fails it as well (and this is already known; see example just after [11, Corollary 2.4]). Moreover, since ℓ_∞^2 is isometrically isomorphic to ℓ_1^2, the pair (ℓ_2^2, ℓ_1^2) also fails the uniform property. Next we show that the pair $(\ell_2^2(\mathbb{R}), \ell_q^2(\mathbb{R}))$ for $1 \leq q < 2$ also fails property 2.

Proposition 3. Let $1 \leq q < 2$. Given $\beta \in (0,1)$, there exists $T_\beta : \ell_2^2 \longrightarrow \ell_q^2$ with $\|T_\beta\| = 1$ such that

(i) $\|T_\beta(e_1)\|_q = \beta$ and
(ii) for every $z \in \ell_2^2$ such that $\|T_\beta(z)\|_q = 1$ we have $\|z - e_1\|_2 = \sqrt{2}$.

Proof. Let $1 \leq q < 2$ and define $T : \ell_2^2 \longrightarrow \ell_q^2$ by

$$T(x,y) := \left(\frac{x-y}{2^{\frac{1}{q}}}, \frac{x+y}{2^{\frac{1}{q}}} \right)$$

for every $(x,y) \in \ell_2^2$. First of all, note that

$$\|T(e_2)\|_q^q = \left| -\frac{1}{2^{\frac{1}{q}}} \right|^q + \left(\frac{1}{2^{\frac{1}{q}}} \right)^q = \frac{1}{2} + \frac{1}{2} = 1,$$

i.e., $\|T(e_2)\|_q = 1$. Analogously, $\|T(e_1)\|_q = 1$. Since T is a scalar multiple of the composition of a rotation in ℓ_2^2 and the identity from ℓ_2^2 into ℓ_q^2, we have that $\|T\| = 1$. Next, we show that

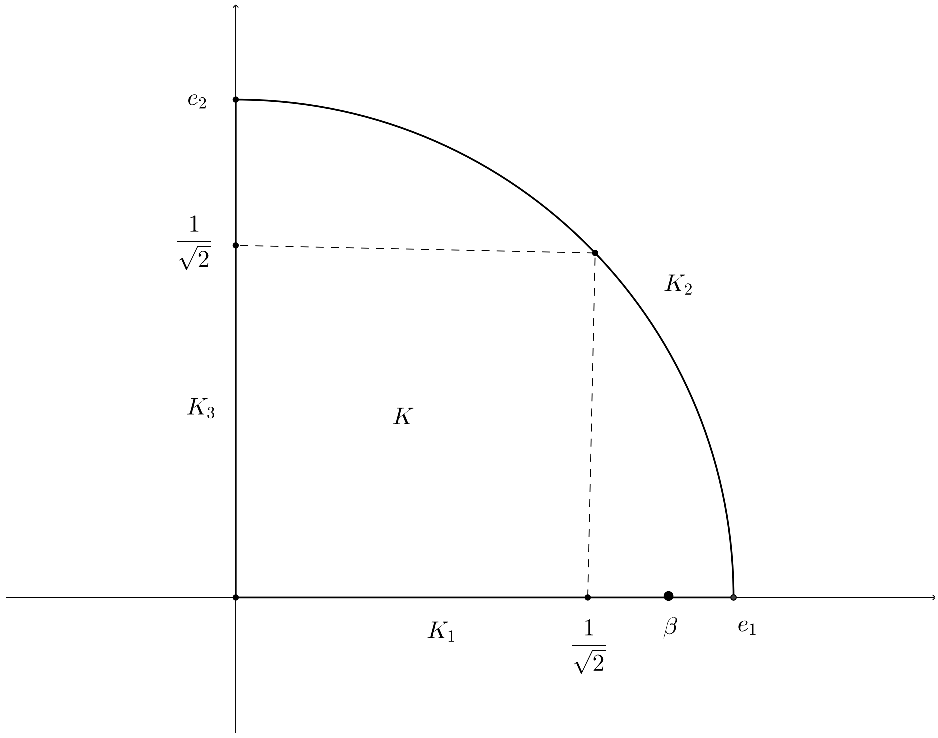

FIGURE 1. The compact set K

the only points which T attains its norm are at $\pm e_1$ and $\pm e_2$. To do so, we study the norm of the operator T by using the following compact set:

$$K := \left\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1,\ a, b \geq 0\right\}.$$

By symmetry, the norm of T is the maximum of $\|T(z)\|$ with z in K. Let $z_0 = (a_0, b_0)$ a point of K such that T attains its norm at z_0, that is, $\|T\| = \|T(z_0)\|$. We consider K_1 as the segment that connect $(0, 0)$ with e_1, K_3 as the segment that connect $(0, 0)$ with e_2 and K_2 as the arc that connect e_1 with e_2. See Figure 1.

It is enough to study the values of $\|T(z)\|_q$ on the set $K_2 \setminus \{e_1, e_2\}$ since the operator T attains its norm at elements of the sphere and $\|T(e_1)\|_q = \|T(e_2)\|_q = 1$. We have

$$K_2 \setminus \{e_1, e_2\} = \left\{(x, f(x)) : x \in \left(\frac{1}{\sqrt{2}}, 1\right)\right\} \cup \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\} \cup \left\{(g(y), y) : y \in \left(\frac{1}{\sqrt{2}}, 1\right)\right\},$$

with $f : \left(\frac{1}{\sqrt{2}}, 1\right) \to \mathbb{R}$ defined as $f(x) = (1 - x^2)^{\frac{1}{2}}$ and $g : \left(\frac{1}{\sqrt{2}}, 1\right) \to \mathbb{R}$ defined as $g(y) = (1 - y^2)^{\frac{1}{2}}$. Since

$$\left\|T\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\|_q = \frac{2}{2^{\frac{1}{2}+\frac{1}{q}}} < 1$$

for every $1 \leq q < 2$, then $z_0 \neq \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. (On the other hand observe that if $q = 2$, then $\left\|T(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\right\|_q = 1$ and if $q > 2$, then $\left\|T(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\right\|_q > 1$.) Thus if $z_0 \in K_2 \setminus \{e_1, e_2\}$, then either

$z_0 \in \left\{(x, f(x)) : x \in \left(\frac{1}{\sqrt{2}}, 1\right)\right\}$ and then a_0 would be a critical point of F in $\left(\frac{1}{\sqrt{2}}, 1\right)$, where

$$F(x) = \|T(x, f(x))\|_q^q = \frac{1}{2}\left[\left(x - (1-x^2)^{\frac{1}{2}}\right)^q + \left(x + (1-x^2)^{\frac{1}{2}})\right)^q\right.$$

or $z_0 \in \left\{(g(y), y) : y \in \left(\frac{1}{\sqrt{2}}, 1\right)\right\}$ and in this case b_0 would be a critical point of G in $\left(\frac{1}{\sqrt{2}}, 1\right)$, where

$$G(y) = \|T(g(y), y)\|_q^q = \frac{1}{2}\left[\left(y - (1-y^2)^{\frac{1}{2}}\right)^q + \left((1-y^2)^{\frac{1}{2}} + y\right)^q\right].$$

But, as we will see in the next lines, these can not happen because $F'(x) > 0$ and $G'(y) > 0$ for all $x, y \in \left(\frac{1}{\sqrt{2}}, 1\right)$ and then $z_0 \notin K_2 \setminus \{e_1, e_2\}$. Indeed, we consider first the case that $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$ For every $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$, we get

$$F'(x) = \frac{q}{2}\left[\left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1}\left(1 + \frac{x}{(1-x^2)^{\frac{1}{2}}}\right) + \left(x + (1-x^2)^{\frac{1}{2}}\right)^{q-1}\left(1 - \frac{x}{(1-x^2)^{\frac{1}{2}}}\right)\right].$$

For $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$, we have that $\left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1} > 0$ and since

$$\left(x + (1-x^2)^{\frac{1}{2}}\right)^{q-1} \geq \left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1}$$

for every x on this interval, we obtain that

$$F'(x) \geq \frac{q}{2}\left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1}\left(1 + \frac{x}{(1-x^2)^{\frac{1}{2}}} + 1 - \frac{x}{(1-x^2)^{\frac{1}{2}}}\right) = q\left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1} > 0,$$

for every $x \in (\frac{1}{\sqrt{2}}, 1)$. A simply change of the letter F by G and x by y implies that $G'(y) > 0$ for every $y \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

Everything we did so far was to prove that T attains its norm on K only at $z = e_1$ and $z = e_2$. Therefore, we may conclude that T attains its maximum at $\pm e_1$ and at $\pm e_2$. In other words, we proved that $T(B_{\ell_2^2}) \cap S_{\ell_q^2} = \{\pm e_1, \pm e_2\}$.

Now, for $0 < \beta < 1$, define $T_\beta : \ell_2^2 \longrightarrow \ell_q^2$ by

$$T_\beta(x, y) = \left(\frac{\beta x - y}{2^{\frac{1}{q}}}, \frac{\beta x + y}{2^{\frac{1}{q}}}\right),$$

for every $(x, y) \in \ell_2^2$. Note that $\|T_\beta(e_1)\|_q = \beta$ and $\|T_\beta(e_2)\|_q = 1$. Since $T_\beta(B_{\ell_2^2}) \subset T(B_{\ell_2^2}) \subset B_{\ell_q^2}$, then $\|T_\beta\| \leq 1$. Also, using that $T(B_{\ell_2^2}) \cap S_{\ell_q^2} = \{\pm e_1, \pm e_2\}$ and that $\|T_\beta(\pm e_1)\|_q < 1$, we have that $\|T_\beta(\pm e_2)\|_q = 1$. This implies that if $z \in S_{\ell_2^2}$ is such that $\|T_\beta(z)\|_q = 1$, then $z = \pm e_2$ and therefore $\|e_1 - z\|_2 = \sqrt{2}$ as we wanted.

□

As a consequence of Propositions 2 and 3 we have the following corollary.

Corollary 4. The pair $(\ell_2^2(\mathbb{R}), \ell_q^2(\mathbb{R}))$ fails property 2 for every $1 \leq q \leq \infty$.

What about the case that $1 < p \leq 2$ and $1 \leq q < 2$? We will study the real case of this right now. Consider $1 < p \leq 2$. Define $Id : \ell_p^2(\mathbb{R}) \to \ell_2^2(\mathbb{R})$ by $Id(x,y) = (x,y)$ for every $(x,y) \in \ell_p^2(\mathbb{R})$. Then $Id(e_1) = e_1$ and $Id(e_2) = e_2$. Since $p \leq 2$, it is clear that $\|Id\| = 1$. Given $0 < \beta < 1$, let $T_\beta : \ell_2^2(\mathbb{R}) \longrightarrow \ell_q^2(\mathbb{R})$ be as in the Proposition 3 with $1 \leq q < 2$. Now, define $\widetilde{T}_\beta : \ell_p^2(\mathbb{R}) \longrightarrow \ell_q^2(\mathbb{R})$ by $\widetilde{T}_\beta = T_\beta \circ Id$. Then $\|\widetilde{T}_\beta\| \leq \|T_\beta\|\|Id\| = 1$. Also, $\|\widetilde{T}_\beta(e_1)\|_q = \|(T_\beta \circ Id)(e_1)\|_q = \beta$ and $\|\widetilde{T}_\beta(e_2)\|_q = \|(T_\beta \circ Id)(e_2)\|_q = \|T_\beta(e_2)\|_q = 1$. Suppose that there exists $z \in S_{\ell_p^2(\mathbb{R})}$ such that $\|\widetilde{T}_\beta(z)\|_q = 1$. Then $\|T_\beta(z)\|_q = 1$ and then, as we can see in the proof of the Proposition 3, z must be equals to e_2 or $-e_2$. In both cases, we have that $\|e_1 - z\|_p = 2^{\frac{1}{p}}$. We just have proved the following result.

Corollary 5. The pair $(\ell_p^2(\mathbb{R}), \ell_q^2(\mathbb{R}))$ fails property 2 for $1 < p \leq 2$ and $1 \leq q \leq 2$.

Next we observe that whenever we put the surpremum norm in the range space, the property fails for any pair of the form (X, ℓ_∞^2).

Proposition 4. The pair (X, ℓ_∞^2) fails property 2 for all Banach space X with $\dim(X) \geq 2$.

Proof. Suppose that there exists $\eta(\varepsilon) > 0$ that depends only on a given $\varepsilon > 0$ satisfying the property. Let $x_1^*, x_2^* \in S_{X^*}$ and $x_1, x_2 \in S_X$ be such that $x_i^*(x_j) = \delta_{ij}$ for $i,j = 1,2$. Define $T : X \longrightarrow \ell_\infty^2$ by

$$T(x) := ((1 - \eta(\varepsilon))\, x_1^*(x), x_2^*(x)),$$

for all $x \in X$. Then $\|T(x_1)\|_\infty = 1 - \eta(\varepsilon)$ and $\|T(x_2)\|_\infty = 1$. Moreover, since $1 - \eta(\varepsilon) < 1$, we have that $\|T\| \leq 1$. This shows that $\|T\| = 1$. Therefore, there exists $z \in S_X$ such that

$$\|T(z)\|_\infty = 1 \quad \text{and} \quad \|z - x_1\| < \varepsilon.$$

Since $\|T(z)\|_\infty = \max\{|(1 - \eta(\varepsilon))|x_1^*(z)|, |x_2^*(z)|\}$ and $(1 - \eta(\varepsilon))|x_1^*(z)| < 1$, we have that $|x_2^*(z)| = 1$. On the other hand, since $|x_2^*(z - x_1)| \leq \|z - x_1\| < \varepsilon$ we get a contradiction, since

$$1 = |x_2^*(z)| = |x_2^*(z - x_1) + x_2^*(x_1)| = |x_2^*(z - x_1)| < \varepsilon < 1.$$

□

We show now that if Y is a 2-dimensional Banach space, then the pair (Y, Y) does not have property 2. To do so, we use the existence of the Auerbach basis for a finite dimensional Banach space (see, for example, [8, Proposition 20.21]). Let Y be an n-dimensional Banach space. Then there are elements $e_1, \ldots, e_n$ of Y and $y_1^*, \ldots, y_n^*$ of Y^* such that $\|e_i\| = \|y_i^*\| = 1$ for all i and $y_i^*(e_j) = \delta_{ij}$ for $i \neq j$. In fact, $\{e_1, \ldots, e_n\}$ is a basis of Y called the *Auerbach basis* of Y.

Proposition 5. Let Y be a 2-dimensional Banach space. Then the pair (Y, Y) fails property 2.

Proof. We start by taking an Auerbach basis: let $\{e_1, e_2\}$ and $\{y_1^*, y_2^*\}$ satisfying $\|e_i\| = \|y_i^*\| = 1$ for $i = 1,2$ and $y_i^*(e_j) = \delta_{ij}$ for $i,j = 1,2$. Since $\{e_1, e_2\}$ is a basis for Y, every $y \in Y$ has an expression in terms of e_1, e_2, y_1^* and y_2^* given by $y = y_1^*(y)e_1 + y_2^*(y)e_2$. Given $\beta \in (0,1)$, define the continuous linear operator $T_\beta : Y \to Y$ by $T_\beta(y) = \beta y_1^*(y)e_1 + y_2^*(y)e_2$ for all

$y = y_1^*(y)e_1 + y_2^*(y)e_2 \in Y$. Then for all $y \in S_Y$, we have that

$$\begin{aligned} \|T_\beta(y)\| &= \|\beta y_1^*(y)e_1 + y_2^*(y)e_2\| \\ &= \|\beta y_1^*(y)e_1 + \beta y_2^*(y)e_2 - \beta y_2^*(y)e_2 + y_2^*(y)e_2\| \\ &\leq \|\beta(y_1^*(y)e_1 + y_2^*(y)e_2)\| + \|(1-\beta)y_2^*(y)e_2\| \\ &= \beta\|y\| + (1-\beta)\|y_2^*(y)e_2\| \\ &\leq \beta\|y\| + (1-\beta)|y_2^*(y)| \\ &\leq \beta + 1 - \beta \\ &= 1. \end{aligned}$$

Then $\|T_\beta\| \leq 1$. Also, note that $\|T_\beta(e_2)\| = \|e_2\| = 1$. So $\|T_\beta\| = 1$. Now let $y_0 \in S_Y$ be such that $\|T_\beta(y_0)\| = 1$. Then, using that

$$1 = \|T_\beta(y_0)\| \leq \beta\|y_0\| + (1-\beta)|y_2^*(y_0)| \leq 1,$$

we get that $|y_2^*(y_0)| = 1$ and therefore $\|e_1 - y_0\| \geq |y_2^*(e_1) - y_2^*(y_0)| = |-1| = 1$. Finally, if the pair (Y, Y) has property 2, there exists $\eta(\varepsilon) > 0$ satisfying the property. If we put $\beta = 1 - \frac{\eta(\varepsilon)}{2}$, there exists an operator $T \in \mathcal{L}(Y, Y)$ such that $\|T\| = 1$, $\|T(e_1)\| > 1 - \eta(\varepsilon)$ and for all $y_0 \in S_Y$ which satisfies $\|T(y_0)\| = 1$ is such that $\|e_1 - y_0\| \geq 1$. This is a contradiction and the pair (Y, Y) fails property 2. □

Remark 2. If $\dim(Y) = n \geq 2$, the proof of Proposition 5 works as well in this situation. Indeed, for $\beta \in (0, 1)$ we define $T_\beta \in \mathcal{L}(Y, Y)$ by

$$T_\beta(y) = \beta y_1^*(y)e_1 + \beta y_2^*(y)e_2 + \ldots + \beta y_{n-1}^*(y)e_{n-1} + y_n^*(y)e_n$$

for all $y \in Y$, where $\{e_1, \ldots, e_n\} \subset S_Y$ and $\{y_1^*, \ldots, y_n^*\} \subset S_{Y^*}$ is given by the Auerbach basis. Then $\|T_\beta(e_i)\| = \beta$ for $i \neq n$ and $\|T_\beta(e_n)\| = 1$. To prove that $\|T_\beta\| \leq 1$, we add and subtract $\beta y_1^*(y)e_1 + \ldots y_{n-2}^*(y)e_{n-2} + \beta y_n^*(y)e_n$ in $\|T_\beta(y)\|$ where $y \in S_Y$, to get $\|T_\beta(y)\| \leq \beta\|y\| + (1-\beta)|y_n^*(y)| \leq 1$. Now, if T_β attains its norm at some $y_0 \in S_Y$ then $\|e_i - y_0\| \geq |y_n^*(e_i - y_0)| = |y_n^*(y_0)| = 1$ for all $i \neq n$.

It is clear but it is worth to mention that if the pair (X, Y) has property 2, then the pair (X, Z) also has this property for all closed subspace Z of Y. Because of that since ℓ_∞^2 is a closed subspace of $C[0, 1]$ and the pair $(\ell_2^2, \ell_\infty^2)$ does not satisfy property 2 (see Example 1), then the pair $(\ell_2^2, C[0, 1])$ also fails this property. Another consequence of this fact is given in the next corollary.

Corollary 6. If Y is a Banach space which contains strictly convex 2-dimensional subspaces, then there exists a uniformly convex Banach space X such that the pair (X, Y) fails property 2.

Proof. Indeed, let Z be a subspace of Y such that Z is stricly convex and $\dim(Z) = 2$. Then $X = Z$ is uniformly convex, since Z is finite dimensional. By Proposition 5, the pair (X, Z) fails property 2 and by the above observation the pair (X, Y) can not have this property. □

By Corollary 2, the pair (ℓ_2, Z) has property 1 if $\dim(Z) < \infty$. But in the case of property 2, we get a negative result. In fact, we show in the next remark that the pair (ℓ_2, ℓ_2^2) fails property 2.

Remark 3. Suppose by contradiction that the pair (ℓ_2, ℓ_2^2) satisfies property 2. Then given $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(\ell_2, \ell_2^2)}$ and $x_0 \in S_{\ell_2}$ are such that $\|T(x_0)\|_2 > 1 - \eta(\varepsilon)$, there is $x_1 \in S_{\ell_2}$ such that $\|T(x_1)\| = 1$ and $\|x_1 - x_0\| < \varepsilon$. Since the pair (ℓ_2^2, ℓ_2^2) fails property 2, there exists some $\varepsilon_0 > 0$, a norm one linear operator $R : \ell_2^2 \longrightarrow \ell_2^2$ and a norm one vector $(a_0, b_0) \in S_{\ell_2^2}$ with $\|R(a_0, b_0)\| > 1 - \eta(\varepsilon_0)$ such that there is no point $(c_1, c_2) \in S_{\ell_2^2}$ such that $\|R(c_1, c_2)\|_2 = 1$ and $\|(c_1, c_2) - (a_0, b_0)\|_2 < \varepsilon_0$. Let $\pi : \ell_2 \longrightarrow \ell_2^2$ be the projection on the first two coordinates, i.e., $\pi((a_n)_n) := (a_1, a_2)$ for all $(a_n)_n \in \ell_2$. Then $\|\pi\| = 1$. Define $T : \ell_2 \longrightarrow \ell_2^2$ by $T := R \circ \pi$. Then $\|T\| = \|R\| = 1$. Let $x_0 := (a_0, b_0, 0, 0, \dots) \in S_{\ell_2}$. We have that

$$\|T(x_0)\| = \|R(a_0, b_0)\| > 1 - \eta(\varepsilon_0).$$

Then there exists $x_1 := (c_n)_n \in S_{\ell_2}$ such that $\|T(x_1)\|_2 = 1$ and $\|x_1 - x_0\|_2 < \varepsilon_0$. Since

$$1 = \|T(x_1)\|_2 = \|R(\pi(x_1))\|_2 = \|R(c_1, c_2)\|_2 \leq \|(c_1, c_2)\|_2 \leq \|x_1\|_2 = 1,$$

we get that $\|R(c_1, c_2)\|_2 = \|(c_1, c_2)\|_2 = 1$. On the other hand,

$$\|(a_0, b_0) - (c_1, c_2)\|_2 \leq \|x_0 - x_1\|_2 < \varepsilon_0.$$

This is a contradiction and then the pair (ℓ_2, ℓ_2^2) fails property 2 as desired.

We would like to comment that there exists an infinite-dimensional Banach space X such that the pair (X, ℓ_∞) fails property 1. Indeed, by using the ideas of [14, Lemma 2.2], if X an infinite-dimensional Banach space, then we can construct a norm one bounded linear operator from X into ℓ_∞ which never attains its norm. So the pair (X, ℓ_∞) can not have property 1 since otherwise every operator must attain its norm.

As the last result of this paper, we present a complete characterization for the pairs (ℓ_p, ℓ_q) concerning property 1 by showing that there are cases that these pairs satisfy the property and other cases not. In particular, there are uniformly convex (and then reflexive) Banach spaces X such that the pair (X, Y) fails property 1. To prove item (ii) below we will use similar property 2's counterexamples techniques.

Theorem 5. The following holds.

(i) The pair (ℓ_p, ℓ_q) has property 1 whenever $1 \leq q < p < \infty$.
(ii) The pair (ℓ_p, ℓ_q) fails property 1 whenever $1 < p \leq q < \infty$.

Proof. (i) By Pitt theorem (see, for example, [2, Theorem 2.1.4]) every bounded linear operator from ℓ_p into ℓ_q with $1 \leq q < p < \infty$ is compact. By Theorem 4 the pair (ℓ_p, ℓ_q) has property 1 since ℓ_p is uniformly convex for $1 < p < \infty$.

(ii) Let $\varepsilon \in (0, 1)$ and $1 < p \leq q < \infty$. Consider ℓ_p and ℓ_q as the Banach spaces $\ell_p(\ell_p^2)$ and $\ell_q(\ell_q^2)$, respectively. For each $n \in \mathbb{N}$ define $T_n : \ell_p^2 \longrightarrow \ell_q^2$ by

$$T_n(x, y) := \left(\left(1 - \frac{1}{2n} \right) x, y \right)$$

for each $(x, y) \in \ell_p^2$. Let $z = ((x_n, y_n))_{n \in \mathbb{N}} \in \ell_2$ and let $T : \ell_p \longrightarrow \ell_q$ be defined by

$$T(z) := (T_n(x_n, y_n))_{n \in \mathbb{N}} = \left(\left(1 - \frac{1}{2n} \right) x_n, y_n \right)_{n \in \mathbb{N}}.$$

For each $z = ((x_n, y_n))_{n\in\mathbb{N}} \in \ell_p$ we have that $\|z\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p + |y_j|^p\right)^{\frac{1}{p}}$ and then

$$\|T(z)\|_q = \left(\sum_{j=1}^{\infty} \left(1 - \frac{1}{2j}\right)^q |x_j|^q + |y_j|^q\right)^{\frac{1}{q}} \leq \left(\sum_{j=1}^{\infty} |x_j|^q + |y_j|^q\right)^{\frac{1}{q}} = \|z\|_q \leq \|z\|_p.$$

So $\|T\| \leq 1$. We consider the vectors

$$e_{1,n} := ((0,0), \ldots, (0,0), (1,0), (0,0), \ldots) \text{ and } e_{2,n} := ((0,0), \ldots, (0,0), (0,1), (0,0), \ldots) \in S_{\ell_p}.$$

Thus we get that $\|T(e_{2,n})\|_q = \|(0,1)\|_q = 1$. So $\|T\| = 1$. Suppose that there exists $\eta(\varepsilon, T) > 0$ such that the pair (ℓ_p, ℓ_q) has property 1. Let $n \in \mathbb{N}$ be such that $\frac{1}{2n} < \eta(\varepsilon, T)$. So since $\|T\| = \|e_{1,n}\|_p = 1$ and

$$\|T(e_{1,n})\|_q = 1 - \frac{1}{2n} > 1 - \eta(\varepsilon, T)$$

there exists $v = (u_n, w_n) \in \ell_p$ such that

$$\|T(v)\|_q = \|v\|_p = 1 \quad \text{and} \quad \|v - e_{1,n}\|_2 < \varepsilon.$$

Next we claim that $u_j = 0$ for all $j \in \mathbb{N}$. Indeed, suppose that there exists some $j_0 \in \mathbb{N}$ such that $u_{j_0} \neq 0$. Thus

$$\begin{aligned}
\|T(v)\|_q &= \left(\sum_{j=1}^{\infty} \left(1 - \frac{1}{2j}\right)^q |u_j|^q + |w_j|^q\right)^{\frac{1}{q}} \\
&= \left(\left(1 - \frac{1}{2j_0}\right)^q |u_{j_0}|^q + |w_{j_0}|^q + \sum_{j\neq j_0} \left(1 - \frac{1}{2j}\right)^q |u_j|^q + |w_j|^q\right)^{\frac{1}{q}} \\
&< \left(|u_{j_0}|^q + |w_{j_0}|^q + \sum_{j\neq j_0} |u_j|^q + |w_j|^q\right)^{\frac{1}{q}} \\
&= \|v\|_q \\
&\leq \|v\|_p = 1
\end{aligned}$$

which is a contradiction. Then $u_j = 0$ for all $j \in \mathbb{N}$. Because of that, we have

$$\|e_{1,n} - v\|_q = \left(1 + \sum_{j\neq n} |w_j|^q\right)^{\frac{1}{q}} \geq 1^{\frac{1}{q}} = 1 > \varepsilon.$$

This new contradiction shows that the pair (ℓ_p, ℓ_q) fails property 1 whenever $1 < p \leq q < \infty$.

□

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