ON THE BISHOP-PHELPS-BOLLOBÁS THEOREM FOR MULTILINEAR MAPPINGS

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Abstract. We study the Bishop-Phelps-Bollobás property and the Bishop-Phelps-Bollobás property for numerical radius. Our main aim is to extend some known results about norm or numerical radius attaining operators to multilinear and polynomial cases. We characterize the pair $(\ell_1(X), Y)$ to have the BPBp for bilinear forms and prove that on $L_1(\mu)$ the numerical radius and the norm of a multilinear mapping are the same. To do so, we prove a result that relates the numerical radius on a direct sum to the numerical radius of its components and that $L_1(\mu)$ fails the BPBp-nu for multilinear mappings although $L_1(\mu)$ satisfies it in the operator case for every measure μ.

1. Introduction

This paper was motivated by the Bishop-Phelps-Bollobás theorem [11]. This theorem says that if X is a Banach space and $\varepsilon > 0$, there exists a positive real number $\eta > 0$ which depends only on ε such that whenever $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ satisfy $|x_0^*(x_0)| > 1 - \eta$, there are $x_1 \in S_X$ and $x_1^* \in S_{X^*}$ such that $|x_1^*(x_1)| = 1$, $\|x_1 - x_0\| < \varepsilon$ and $\|x_1^* - x_0^*\| < \varepsilon$. In particular, this gives the Bishop-Phelps theorem which says that the set $\mathrm{NA}(X)$ of all norm attaining functionals is dense in X^* [14]. In 1963, Lindenstrauss showed that there is no version of the Bishop-Phelps theorem (and consequently there is no version of the Bishop-Phelps-Bollobás theorem) for bounded linear operators [30]. On the other hand, he gave positive results by putting conditions on the domain and range spaces.

Eight years ago, Acosta, Aron, García and Maestre started a similar work with the Bishop-Phelps-Bollobás theorem [1]. A pair $(X;Y)$ of Banach spaces has the *Bishop-Phelps-Bollobás property* (which is referred simply as the BPBp) when given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X;Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy $\|T(x_0)\| > 1 - \eta(\varepsilon)$, there are $S \in \mathcal{L}(X;Y)$ with $\|S\| = 1$ and $x_1 \in S_X$ such that $\|S(x_1)\| = 1$, $\|x_1 - x_0\| < \varepsilon$ and $\|S - T\| < \varepsilon$ [1, Definition 1.1]. In this case, we say that the pair $(X;Y)$ has the BPBp with function $\varepsilon \longmapsto \eta(\varepsilon)$ and when $Y = \mathbb{K}$ we just say that X has the BPBp (note that this is just the Bishop-Phelps-Bollobás theorem). The authors showed, for example, that the pair $(X;Y)$ has this property whenever X and Y are finite-dimensional. In case that Y has the property β of Lindenstrauss, the pair $(X;Y)$ also has the BPBp for any Banach space X. They also gave a characterization for the pair $(\ell_1;Y)$ via the geometry of the Banach space Y. More precisely, the pair $(\ell_1;Y)$ has the BPBp if and only if Y has the approximate hyperplane series property (AHSP, for short). They proved that finite-dimensional Banach spaces, $L_1(\mu)$-spaces, $C(K)$-spaces and uniformly convex Banach spaces satisfy this property [1, Proposition 3.5, 3.6, 3.7 and 3.8, respectively]. Later, Kim, Lee and Martín defined a stronger property [28, see Definition 4] and they called it as the *generalized AHSP*. They observed that this property implies the BPBp for the pair $(X;Y)$ and they gave an analogous characterization for the pair $(\ell_1(X);Y)$.

Inspired by the Bishop-Phelps-Bollobás property, some authors studied the Bishop-Phelps-Bollobás property for numerical radius (see [10, 23, 25, 27]). We recall the concept of numerical radius to give the definition. We denote by $\Pi(X)$ the set of all pairs $(x, x^*) \in S_X \times S_{X^*}$ such that $x^*(x) = 1$. Given a bounded linear operator $T : X \longrightarrow X$, we define its *numerical radius* by

$$v(T) := \sup\{|x^*(T(x))| : (x, x^*) \in \Pi(X)\}.$$

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It is not difficult to see that v is a semi-norm on the Banach space $\mathcal{L}(X)$ of all bounded linear operators from X into X. The inequality $v(T) \leq \|T\|$ always holds for all $T \in \mathcal{L}(X)$. We refer the reader to [12, 13] for more information and background about numerical radius theory. We say that a Banach space X has the *Bishop-Phelps-Bollobás property for numerical radius* (BPBp-nu, for short) if for every $\varepsilon > 0$, there exists some $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X;X)$ with $v(T) = 1$ and $(x, x^*) \in \Pi(X)$ satisfy $|x^*(T(x))| > 1 - \eta(\varepsilon)$, there are $S \in \mathcal{L}(X)$ with $v(S) = 1$ and $(y, y^*) \in \Pi(X)$ such that $|y^*(S(y))| = 1$, $\|y^* - x^*\| < \varepsilon$, $\|y - x\| < \varepsilon$ and $\|S - T\| < \varepsilon$. The Banach spaces ℓ_1 and c_0 have the BPBp-nu ([25, Corollary 3.3 and Corollary 4.2] as well as the finite dimensional Banach spaces [27, Proposition 2] and the Banach space $L_1(\mu)$ for every measure μ [27, Theorem 4.1] (see also [23]). Moreover, the Banach space $C(K)$ has this property in some cases [10, Theorem 2.2]. It is known also that the L_p-spaces satisfy the BPBp-nu when $1 < p < \infty$ in the complex case and when $1 < p < \infty$ with $p \neq 2$ in the real case [27, Examples 3.5].

In this article, we study the BPBp and the BPBp-nu in the multilinear vein. To do so, we define the objects and tools that we need from now on. Next, we summarize our main results.

Let $X_1, \ldots, X_N$ and Y be Banach spaces. We denote by $\mathcal{L}(X_1, \ldots, X_N; Y)$ the set of all bounded N-linear mappings defined from $X_1 \times \ldots \times X_N$ into Y. We use the letters A, B, C or D to denote members of $\mathcal{L}(X_1, \ldots, X_N; Y)$. If $A \in \mathcal{L}(X_1, \ldots, X_N; Y)$, then we define the norm of A by

$$\|A\| := \sup\left\{\|A(x_1, \ldots, x_N)\| : (x_1, \ldots, x_N) \in S_{X_1} \times \ldots \times S_{X_N}\right\}.$$

We say that $A \in \mathcal{L}(X_1, \ldots, X_N; Y)$ *attains its norm* or it is *norm attaining* when there exists some point $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ such that $\|A(x_1^0, \ldots, x_N^0)\| = \|A\|$. We denote by $\mathrm{NA}(\mathcal{L}(X_1, \ldots, X_N; Y))$ the set of all norm attaining multilinear mappings. When $X_1 = \ldots = X_N = X$, we write $\mathcal{L}(^N X; Y)$. We say that $A \in \mathcal{L}(^N X)$ is *symmetric* whenever $A(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = A(x_1, \ldots, x_N)$ for all $(x_1, \ldots, x_N) \in X \times \ldots \times X$ and every permutation σ on $\{1, \ldots, N\}$. We denote by $\mathcal{L}_s(^N X; Y)$ the set of all symmetric multilinear mappings from $X \times \ldots \times X$ into Y. A mapping $P : X \longrightarrow Y$ is said to be an N*-homogeneous polynomial* if there exists some symmetric N-linear mapping $\hat{A} \in \mathcal{L}_s(^N X; Y)$ such that $P(x) = \hat{A}(x, \ldots, x)$ for all $x \in X$. We denote by $\mathcal{P}(^N X; Y)$ the set of all continuous N-homogeneous polynomials from X into Y. This space is a Banach space equipped with the usual norm given by $\|P\| := \sup_{x \in S_X} \|P(x)\|$ for $P \in \mathcal{P}(^N X; Y)$. We say that an N-homogeneous polynomial P *attains its norm* or it is *norm attaining* if there exists some $x_0 \in S_X$ such that $\|P(x_0)\| = \|P\|$. We denote by $\mathrm{NA}(\mathcal{P}(^N X; Y))$ the set of all norm attaining N-homogeneous polynomials.

In the next section, we study the Bishop-Phelps-Bollobás property for multilinear mappings and homogeneous polynomials (go to this section to see the proper definitions). We extend some known results about norm attaining multilinear mappings to the BPBp as a stability result which says that we can pass from $(N+1)$-degree to a N-degree in the BPBp. We prove that if Y has property β (see the definition below), then the N-tuple $(X_1, \ldots, X_N)$ has the BPBp if and only if $(X_1, \ldots, X_N; Y)$ has the BPBp. We also study the BPBp for compact multilinear mappings and and provide examples satisfying such property.

The main goal of Section 3 is to characterize the pair $(\ell_1(X), Y)$ to have BPBp for bilinear forms. In order to do this, we study the generalized approximate hyperplane series property (generalized AHSP) for bilinear forms.

Section 4 is dedicated to the study of the numerical radius on the set of all multilinear mappings defined in $L_1(\mu)$, where μ is an arbitrary measure. We prove that for every $A \in \mathcal{L}(^N L_1(\mu); L_1(\mu))$, its numerical radius and its norm coincide. To do this, we prove a result that relates the numerical radius on a direct sums with the numerical radius of each one of its components.

We study the Bishop-Phelps-Bollobás property for numerical radius for multilinear mappings in Section 5. It is shown that if X is a finite-dimensional Banach space, then X satisfies this property. On the other hand, $L_1(\mu)$ fails it although $L_1(\mu)$ has it in the operator case for every measure μ. We also prove that if a c_0 or a ℓ_1-sum satisfies it, then each component of the direct sum also satisfies it.

The results of this paper are valid for real or complex Banach spaces, unless stated otherwise.

2. The BPBp for multilinear mappings

In this section we extend some known results about norm attaining operators and the Bishop-Phelps-Bollobás property to the multilinear and polynomial cases. We start by defining the BPBp for this type of functions.

Definition 2.1. Let $X_1,\dots,X_N$ and Y be Banach spaces. We say that $(X_1,\dots,X_N;Y)$ has the *Bishop-Phelps-Bollobás property for multilinear mappings* (BPBp for multilinear mappings, for short) if given $\varepsilon>0$, there exists $\eta(\varepsilon)>0$ such that whenever $A\in\mathcal{L}(X_1,\dots,X_N;Y)$ with $\|A\|=1$ and $\left(x_1^0,\dots,x_N^0\right)\in S_{X_1}\times\ldots\times S_{X_N}$ satisfy

$$\left\|A\left(x_1^0,\dots,x_N^0\right)\right\|>1-\eta(\varepsilon),$$

there are $B\in\mathcal{L}(X_1,\dots,X_N;Y)$ with $\|B\|=1$ and $\left(z_1^0,\dots,z_N^0\right)\in S_{X_1}\times\ldots\times S_{X_N}$ such that

$$\left\|B\left(z_1^0,\dots,z_N^0\right)\right\|=1,\quad \max_{1\le j\le N}\|z_j^0-x_j^0\|<\varepsilon\quad\text{and}\quad \|B-A\|<\varepsilon. \tag{2.1}$$

In this case, we say that $(X_1,\dots,X_N;Y)$ has the BPBp for multilinear mappings with function $\varepsilon\longmapsto\eta(\varepsilon)$.

When it is of interest we can emphasize the degree of the multilinear mapping by saying that $(X_1,\dots,X_N;Y)$ has the *BPBp for N-linear mappings* instead of the BPBp for multilinear mappings. We may also define the *BPBp for symmetric multilinear mappings* when in the Definition 2.1 we consider A and B both elements in $\mathcal{L}_s(^NX;Y)$. In this case, we say that $(^NX;Y)$ has the BPBp for symmetric multilinear mappings. When $Y=\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we denote the BPBp for $(X_1,\dots,X_N;\mathbb{K})$ just by $(X_1,\dots,X_N)$. Analogously, we define the BPBp for homogeneous polynomials as follows.

Definition 2.2. Let X and Y be Banach spaces. We say that the pair $(X;Y)$ has the *Bishop-Phelps-Bollobás property for N-homogeneous polynomials* if given $\varepsilon>0$, there exists $\eta(\varepsilon)>0$ such that whenever $P\in\mathcal{P}(^NX;Y)$ with $\|P\|=1$ and $x_0\in S_X$ satisfy

$$\|P(x_0)\|>1-\eta(\varepsilon),$$

there are $Q\in\mathcal{P}(^NX;Y)$ with $\|Q\|=1$ and $x_1\in S_X$ such that

$$\|Q(x_1)\|=1,\quad \|x_1-x_0\|<\varepsilon\quad\text{and}\quad\|Q-P\|<\varepsilon.$$

It is worth to mention (and we will use this fact without any explicit reference) that by using a routinely change of parameters in Definitions 2.1 and 2.2 we may consider the given elements in the unit ball of their respectively spaces instead of norm-one elements. For example, in the multilinear mapping case, we can say that $(X_1,\dots,X_N;Y)$ has the BPBp for multilinear mappings if given $\varepsilon>0$, there exists $\eta(\varepsilon)>0$ such that whenever $A\in\mathcal{L}(X_1,\dots,X_N;Y)$ with $\|A\|\le1$ and $\left(x_1^0,\dots,x_N^0\right)\in B_{X_1}\times\ldots\times B_{X_N}$ satisfy $\left\|A\left(x_1^0,\dots,x_N^0\right)\right\|>1-\eta(\varepsilon)$, there are $B\in\mathcal{L}(X_1,\dots,X_N;Y)$ with $\|B\|=1$ and $(z_1^0,\dots,z_N^0)\in S_{X_1}\times\ldots\times S_{X_N}$ satisfying the conditions (2.1).

It was shown in [26, Theorem 2 and Corollary 3] that $(C_0(K),C_0(L))$ and (c_0,c_0) have the BPBp for bilinear forms in the complex case for every locally compact Hausdorff topological spaces K and L. On the other hand, $(L_1[0,1],L_1[0,1])$ fails the BPBp for bilinear forms [15, Theorem 3]. The pair (H,H) has the BPBp for symmetric bilinear forms on a Hilbert space H [24, Theorem 3.2 and Theorem 3.4]. In [5, Theorem 2.2] it was shown that if $X_1,\dots,X_N$ are uniformly convex Banach spaces, then $(X_1,\dots X_N;Y)$ has the BPBp for multilinear mappings for any Banach space Y. Also, if X is a uniformly convex Banach space then $(X;Y)$ has the BPBp for N-homogeneous polynomials for every Banach space Y [3, Theorem 3.1].

We can not expect a BPBp version for multilinear mappings of [9, Theorem 3] which says that if X satisfies property α, then the set $\operatorname{NA}\mathcal{L}^N(X)$ is dense in $\mathcal{L}^N(X)$, since a typical example of a Banach space with this property is ℓ_1 and (ℓ_1,ℓ_1) fails the BPBp for bilinear forms [18, Theorem 2]. The same counterexample shows that there is no BPBp version for multilinear mappings of [1, Theorem 2.2], when we are assuming that the range space Y has property β, since $\mathbb{K}$ satisfies it. Although (ℓ_1,ℓ_1) fails the BPBp for bilinear forms, the set $\operatorname{NA}\mathcal{L}(^N\ell_1;Y)$ is dense in $\mathcal{L}(^N\ell_1;Y)$ for every $N\in\mathbb{N}$ and Banach space Y [17, Theorem 2.4(a)]. The same arguments of [18, Theorem 2] prove that (ℓ_1,ℓ_1) does not have the BPBp for symmetric bilinear forms although it is known that the set $\operatorname{NA}\mathcal{L}_s(^N\ell_1;Y)$ is dense in $\mathcal{L}_s(^N\ell_1;Y)$ for every $N\in\mathbb{N}$ and every Banach space Y [17, Theorem 2.4(b)].

It is known that for every finite-dimensional Banach spaces X and Y, the pair $(X;Y)$ has the BPBp. Our first result concerns the analogous version for the multilinear mappings and homogeneous polynomials, and its proof is just an easy modification of [1, Proposition 2.4] to these cases.

Proposition 2.3. *Let $X,X_1,\dots,X_N$ and Y be finite dimensional Banach spaces. Then*

(i) *$(X_1,\dots,X_N;Y)$ has the BPBp for multilinear mappings,*

(ii) $({}^N X;Y)$ *has the BPBp for symmetric multilinear mappings and*
(iii) $(X;Y)$ *has the BPBp for N-homogeneous polynomials.*

Our next result deals with stability of the BPBp for multilinear mappings. In [32, Proposition 2.1] (see also [6, Proposition 3.1]) it was proved that if X is a Banach space and $N \in \mathbb{N}$ is a natural number, then the set $\operatorname{NA}\mathcal{L}({}^N X;Y)$ is dense in $\mathcal{L}({}^N X;Y)$ whenever the set $\operatorname{NA}\mathcal{L}({}^{N+1}X;Y)$ is dense in $\mathcal{L}({}^{N+1}X;Y)$. By using the same argument we show that this result holds for the Bishop-Phelps-Bollobás property.

Proposition 2.4. *Let $X_1, \ldots X_N, X_{N+1}$ and Y be Banach spaces. If $(X_1, \ldots, X_N, X_{N+1}; Y)$ has the BPBp for $(N+1)$-linear mappings, then $(X_1, \ldots, X_N; Y)$ has the BPBp for N-linear mappings.*

Proof. For $\varepsilon \in (0,1)$, let $\eta(\varepsilon) > 0$ be the BPBp constant for the pair $(X_1, \ldots, X_N, X_{N+1}; Y)$. Let $A : X_1 \times \ldots \times X_N \longrightarrow Y$ be an N-linear mapping with $\|A\| = 1$ and $\left(x_1^0, \ldots, x_N^0\right) \in S_{X_1} \times \ldots \times S_{X_N}$ satisfying $\|A\left(x_1^0, \ldots, x_N^0\right)\| > 1 - \eta\left(\frac{\varepsilon}{2}\right)$. Choose $x_{N+1}^0 \in S_{X_{N+1}}$ and $x_{N+1}^* \in S_{X_{N+1}^*}$ such that $x_{N+1}^*\left(x_{N+1}^0\right) = 1$ and define $\widetilde{A} : X_1 \times \ldots \times X_N \times X_{N+1} \longrightarrow Y$ by $\widetilde{A}(x_1, \ldots, x_N, x_{N+1}) := x_{N+1}^*(x_{N+1})A(x_1, \ldots, x_N)$ for every $(x_1, \ldots, x_N, x_{N+1}) \in X_1 \times \ldots \times X_N \times X_{N+1}$. Then we see that $\|\widetilde{A}\| \leq 1$ and $\|\widetilde{A}\left(x_1^0, \ldots, x_N^0, x_{N+1}^0\right)\| > 1 - \eta\left(\frac{\varepsilon}{2}\right)$. Hence, there are $\widetilde{B} \in \mathcal{L}({}^{N+1}X_1, \ldots, X_N, X_{N+1}; Y)$ with $\|\widetilde{B}\| = 1$ and $\left(z_1^0, \ldots, z_N^0, z_{N+1}^0\right) \in S_{X_1} \times \ldots \times S_{X_N} \times S_{X_{N+1}}$ such that

$$\left\|\widetilde{B}\left(z_1^0, \ldots, z_N^0, z_{N+1}^0\right)\right\| = 1, \quad \max_{1 \leq j \leq N+1} \|z_j^0 - x_j^0\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\widetilde{A} - \widetilde{B}\| < \frac{\varepsilon}{2}.$$

It follows that $|x_{N+1}^*\left(z_{N+1}^0\right)| > 1 - \varepsilon > 0$ and we can define $C : X_1 \times \ldots \times X_N \longrightarrow Y$ by

$$C(x_1, \ldots, x_N) := \frac{1}{x_{N+1}^*\left(z_{N+1}^0\right)} \widetilde{B}\left(x_1, \ldots, x_N, z_{N+1}^0\right)$$

for all $(x_1, \ldots, x_N) \in X_1 \times \ldots \times X_N$. Is is not difficult to prove that $\|C\| = \|C(z_1^0, \ldots, z_N^0)\|$ and that $\|C - A\| < \varepsilon$. To finish the proof, we put $B := \frac{C}{\|C\|}$. Therefore, $\|B\| = \left\|B\left(z_1^0, \ldots, z_N^0\right)\right\| = 1$, $\|B - A\| < 2\varepsilon$ and $\max_{1 \leq j \leq N} \left\|z_j^0 - x_j^0\right\| < \varepsilon$. ∎

Note that the converse of Proposition 2.4 is no longer true. Indeed, ℓ_1 has the BPBp by the Bishop-Phelps-Bollobás theorem but the pair (ℓ_1, ℓ_1) fails the BPBp for bilinear forms [18, Theorem 1]. The next result is about stability as well and its proof is just an easy consequence of the natural (isometric) identification between the Banach spaces $\mathcal{L}({}^N X_1, \ldots, X_N; Y)$ and $\mathcal{L}({}^k X_1, \ldots, X_k; \mathcal{L}({}^{N-k}X_{k+1}, \ldots, X_N; Y))$.

Proposition 2.5. *Assume that $N \geq 2$ and $X_1, \ldots, X_N$ and Y are Banach spaces. If the pair $(X_1, \ldots, X_N; Y)$ has the BPBp for N-linear mappings, then the pair $(X_1, \ldots, X_k; \mathcal{L}({}^{N-k}X_{k+1}, \ldots, X_N; Y))$ has the BPBp for k-linear mappings.*

We observe that the converse of Proposition 2.5 is not true in general. If it were true, it were true also for $N = 2$, $k = 1$ and $Y = \mathbb{K}$. But this would imply that if the pair $(X; Y^*)$ has the BPBp for operators then (X, Y) has the BPBp for bilinear forms which is false in general. For this, we take again $X = Y = \ell_1$ and use [1, Theorem 4.1], which gives that the pair $(\ell_1; \ell_\infty)$ has the BPBp for operators, and [18, Theorem 1], which shows that (ℓ_1, ℓ_1) fails the BPBp for bilinear forms.

In [5, Proposition 2.4] or [19, Theorem 1.1], the authors proved that if X is any Banach space and Y is a uniformly convex Banach space, then $(X; Y^*)$ has the BPBp for operators if and only if (X, Y) has the BPBp for bilinear forms. The analogous for multilinear mappings is the following result.

Proposition 2.6. *Assume that $N \geq 2$ and $X_1, \ldots, X_N$ are Banach spaces. If X_N is uniformly convex, then $(X_1, \ldots, X_N)$ has the BPBp for N-linear mappings if and only if $(X_1, \ldots, X_{N-1}; X_N^*)$ has the BPBp for $(N-1)$-linear mappings.*

Proof. If $(X_1, \ldots, X_N)$ has the BPBp for N-linear mappings, then $(X_1, \ldots, X_{N-1}; X_N^*)$ has the BPBp for $(N-1)$-linear mappings by using Proposition 2.5 with $k = N - 1$ and $Y = \mathbb{K}$. Now let $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$

be given. Assume that $(X_1,\ldots,X_{N-1};X_N^*)$ has the BPBp for $(N-1)$-linear mappings with function $\eta(\varepsilon)>0$ and consider $\delta_{X_N}(\varepsilon)>0$ the modulus of convexity of the space X_N. We take $\xi>0$ satisfying that

$$\xi < N\xi < \min\left\{\frac{\delta_{X_N}(\varepsilon)}{2},\varepsilon\right\}$$

and define

$$\eta'(\xi) := \min\left\{\frac{\delta_{X_N}(\xi)}{2},\eta(\xi)\right\}.$$

Let $A\in\mathcal{L}(X_1,\ldots,X_N)$ and $\left(x_1^0,\ldots,x_N^0\right)\in S_{X_1}\times\ldots\times S_{X_N}$ satisfy $\|A\|=1$ and $\left|A\left(x_1^0,\ldots,x_N^0\right)\right|>1-\eta'(\xi)$. By multiplying A by an appropriate scalar with modulus one, we can assume that Re $A\left(x_1^0,\ldots,x_N^0\right)>1-\eta'(\xi)$. Define $\widetilde{A}:X_1\times\ldots\times X_{N-1}\longrightarrow X_N^*$ by $\widetilde{A}(x_1,\ldots,x_{N-1})(x_N):=A(x_1,\ldots,x_{N-1},x_N)$ for all $(x_1,\ldots,x_{N-1})\in X_1\times\ldots\times X_{N-1}$ and $x_N\in X_N$. Then $\|\widetilde{A}\|=\|A\|=1$ and $\|\widetilde{A}\left(x_1^0,\ldots,x_{N-1}^0\right)\|\geq$ Re $A\left(x_1^0,\ldots,x_{N-1}^0,x_N^0\right)>1-\eta'(\xi)$. Hence, there are $\widetilde{B}\in\mathcal{L}(X_1,\ldots,X_{N-1};X_N^*)$ with $\|\widetilde{B}\|=1$ and $(z_1^0,\ldots,z_{N-1}^0)\in S_{X_1}\times\ldots\times S_{X_{N-1}}$ such that $\|\widetilde{B}\left(z_1^0,\ldots,z_{N-1}^0\right)\|=1$, $\max_{1\leq j\leq N-1}\|z_j^0-x_j^0\|<\xi<\varepsilon$ and $\|\widetilde{B}-\widetilde{A}\|<\xi$. Since X_N is reflexive, there is $z_N^0\in S_{X_N}$ such that Re $\widetilde{B}(z_1^0,\ldots,z_{N-1}^0)(z_N^0)=\|\widetilde{B}\left(z_1^0,\ldots,z_{N-1}^0\right)\|=1$. We see that $\|z_N^0-x_N^0\|<\varepsilon$. Indeed,

$$\begin{aligned}\text{Re }\widetilde{B}\left(z_1^0,\ldots,z_{N-1}^0\right)\left(x_N^0\right) &> \text{Re }\widetilde{A}\left(z_1^0,\ldots,z_{N_1-1}^0\right)\left(x_N^0\right)-\xi\\ &\geq \text{Re }A\left(x_1^0,x_2^0,\ldots,x_{N-1}^0,x_N^0\right)-\sum_{j=1}^{N-1}\|x_j^0-z_j^0\|-\xi\\ &> 1-\delta_{X_N}(\varepsilon).\end{aligned}$$

This implies that $\left\|\frac{x_N^0+z_N^0}{2}\right\|>1-\frac{\delta_{X_N}(\varepsilon)}{2}$ and so $\|z_N^0-x_N^0\|<\varepsilon$. An N-linear map $B:X_1\times\ldots\times X_N\longrightarrow\mathbb{K}$ defined by $B(x_1,\ldots,x_N):=\widetilde{B}(x_1,\ldots,x_{N-1})(x_N)$ for all $(x_1,\ldots,x_{N-1},x_N)\in X_1\times\ldots\times X_{N-1}\times X_N$ satisfy that $\left\|B\left(z_1^0,\ldots,z_N^0\right)\right\|=$ Re $\widetilde{B}\left(z_1^0,\ldots,z_{N-1}^0\right)\left(z_N^0\right)=1$ as well as $\max_{1\leq j\leq N}\|z_j^0-x_j^0\|<\varepsilon$ and $\|B-A\|\leq\|\widetilde{B}-\widetilde{A}\|<\xi<\varepsilon$. This shows that $(X_1,\ldots,X_N)$ has the BPBp for N-linear mappings. ■

In the next result we prove that we may pass from the vector-valued case to the scalar-valued case in the Bishop-Phelps-Bollobás property for multilinear mappings.

Proposition 2.7. *Suppose that $X_1,\ldots,X_N$ and Y are Banach spaces and $Y\neq\{0\}$. If $(X_1,\ldots,X_N;Y)$ has the BPBp for N-linear mappings, then $(X_1,\ldots,X_N)$ has the BPBp for N-linear forms.*

Proof. Let $\varepsilon\in(0,1)$ be given and consider the BPBp constant $\eta(\varepsilon)>0$ for $(X_1,\ldots,X_N;Y)$. Assume that $A\in\mathcal{L}(X_1,\ldots,X_N)$ with $\|A\|=1$ and $\left(x_1^0,\ldots,x_N^0\right)\in S_{X_1}\times\ldots\times S_{X_N}$ satisfy Re $A\left(x_1^0,\ldots,x_N^0\right)>1-\eta\left(\frac{\varepsilon}{2}\right)$. Define $\widetilde{A}:X_1\times\ldots\times X_N\longrightarrow Y$ by $\widetilde{A}(x_1,\ldots,x_N):=A(x_1,\ldots,x_N)y_0$ for some $y_0\in S_Y$ and for all $(x_1,\ldots,x_N)\in X_1\times\ldots\times X_N$. Then $\|\widetilde{A}\|=\|A\|=1$ and $\|\widetilde{A}\left(x_1^0,\ldots,x_N^0\right)\|>1-\eta\left(\frac{\varepsilon}{2}\right)$. Hence, there are $\widetilde{B}\in\mathcal{L}(X_1,\ldots,X_N;Y)$ with $\|B\|=1$ and $\left(z_1^0,\ldots,z_N^0\right)\in S_{X_1}\times\ldots\times S_{X_N}$ such that $\|\widetilde{B}\left(z_1^0,\ldots,z_N^0\right)\|=1,\max_{1\leq j\leq N}\left\|z_j^0-x_j^0\right\|<\frac{\varepsilon}{2}<\varepsilon$ and $\|\widetilde{B}-\widetilde{A}\|<\frac{\varepsilon}{2}$. Choose $y_0^*\in S_{Y^*}$ such that $|y_0^*(\widetilde{B}(z_1^0,\ldots,z_N^0))|=1$ and $|y_0^*(y_0)|=y_0^*(y_0)$. Then the N-linear form $B:X_1\times\ldots\times X_N\longrightarrow\mathbb{K}$ defined by $B:=y_0^*\circ\widetilde{B}$ satisfies that

$$\begin{aligned}\|A-B\| &\leq \|y_0^*\circ\widetilde{B}-y_0^*\circ\widetilde{A}\|+\|y_0^*\circ\widetilde{A}-A\|\\ &\leq \frac{\varepsilon}{2}+|1-y_0^*(y_0)|<\varepsilon\end{aligned}$$

since

$$\begin{aligned}|y_0^*(y_0)| &\geq |A(z_1^0,\ldots,z_N^0)y_0^*(y_0)|=|y_0^*\circ\widetilde{A}(z_1^0,\ldots,z_N^0)|\\ &\geq |y_0^*\circ\widetilde{B}(z_1^0,\ldots,z_N^0)|-|y_0^*\circ(\widetilde{B}-\widetilde{A})(z_1^0,\ldots,z_N^0))|>1-\frac{\varepsilon}{2}.\end{aligned}$$

■

It is clear that the converse of the previous proposition is false because of the Lindenstrauss counterexample for the Bishop-Phelps theorem for operators. Nevertheless, we have the following consequence from [3, Proposition 3.3] when the range space has property β of Lindenstrauss [30].

Corollary 2.8. *Let $X_1, \ldots, X_N$ and Y be Banach spaces and $Y \neq \{0\}$. Assume that Y has property β. The N-tuple $(X_1, \ldots, X_N)$ has the BPBp for N-linear forms if and only if $(X_1, \ldots, X_N; Y)$ has the BPBp for N-linear mappings.*

In [4], the BPBp for certain subspaces of $\mathcal{L}(X;Y)$ were studied. One of the subspace which the authors considered is the set of compact operators. According to their definition, we say that the pair $(X;Y)$ has the BPBp for compact operators when in the definition of the BPBp we consider compact operators T and S. They showed that $(\ell_1;Y)$ has BPBp if and only if $(L_1(\mu);Y)$ has BPBp for compact operator when $L_1(\mu)$ is infinite dimensional. In [20], it was proved, among other results, that if the pair $(X;Y)$ has the BPBp for compact operators, then so does the pair $(X;C(K,Y))$ for every compact Hausdorff topological space K [20, Theorem 3.5(c)]. In particular, $(X;C(K))$ has the BPBp for compact operators, a result already proved in [2][Theorem 4.2]. Analogously to the operator case, we say that $(X_1, \ldots, X_N;Y)$ has the *BPBp for compact multilinear mappings* when in the Definition 2.1 we consider A and B as compact multilinear mappings. We say that the N-linear mapping $A : X_1 \times \ldots \times X_N \longrightarrow Y$ is *compact* if $A(B_{X_1} \times \ldots \times B_{X_N})$ is a precompact set in Y. We denote by $\mathcal{K}(X_1, \ldots, X_N;Y)$ the set of all compact N-linear mappings from $X_1 \times \ldots \times X_N$ into Y. For example, by adapting [3, Proposition 3.3], we may show that if $(X_1, \ldots, X_N)$ has the BPBp for multilinear forms, then $(X_1, \ldots, X_N;Y)$ has the BPBp for compact multilinear mappings whenever Y has property β. Another example is when we assume that $X_1, \ldots, X_N$ are uniformly convex Banach spaces: indeed, we can adapt [5, Theorem 2.2] to get that $(X_1, \ldots, X_N;Y)$ has the BPBp for compact multilinear mappings for all Banach space Y. In [2, Theorem 4.2] it was proved that the pair $(X;Y)$ has the BPBp for compact operators for any Banach space X and for a predual of an L_1-space Y. We will prove the analogous result for multilinear mappings. To do so, we use the fact that a predual of an L_1-space has the metric approximation property [31, Theorem 1]. We say that a Banach space Y has the *metric approximation property* if for every compact set $K \subset Y$ and every $\varepsilon > 0$, there exists a finite rank linear operator $F \in \mathcal{F}(Y;Y)$ with $\|F\| \leq 1$ and $\|F(y) - y\| < \varepsilon$ for all $y \in K$.

Theorem 2.9. *Let $X_1, \ldots, X_N$ be Banach spaces and let Y be a predual of an L_1-space. Suppose that the N-tuple $(X_1, \ldots, X_N)$ has the BPBp for multilinear forms. Then $(X_1, \ldots, X_N;Y)$ has the BPBp for compact multilinear mappings.*

Proof. Suppose that $(X_1, \ldots, X_N)$ has the BPBp for multilinear forms. Let $\varepsilon \in (0,1)$ and $m \in \mathbb{N}$. As a consequence of [3, Proposition 3.3], we have that $(X_1, \ldots, X_N; \ell_\infty^m)$ has the BPBp for multilinear mappings with some $\eta(\varepsilon) > 0$. Let $A \in \mathcal{K}(X_1, \ldots, X_N;Y)$ with $\|A\| = 1$ and $\left(x_1^0, \ldots, x_N^0\right) \in S_{X_1} \times \ldots \times S_{X_N}$ be such that

$$\left\|A\left(x_1^0, \ldots, x_N^0\right)\right\| > 1 - \frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right).$$

Since Y has the metric approximation property, there exists a finite rank operator $F : Y \longrightarrow Y$ with $\|F\| \leq 1$ and $\|F(y) - y\| < \min\left\{\frac{\varepsilon}{8}, \frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right)\right\}$ for every $y \in \overline{A(B_{X_1} \times \ldots \times B_{X_N})}$. It follows that $\|FA\| \neq 0$ and then we define $A' := \frac{1}{\|FA\|}FA$. It is clear that $A' \in \mathcal{K}(X_1, \ldots, X_N;Y)$ with $\|A'\| = 1$. Moreover,

$$\|A' - A\| = \left\|\frac{FA}{\|FA\|} - A\right\| \leq |1 - \|FA\|| + \|FA - A\| \leq 2\|FA - A\| < \frac{\varepsilon}{4}$$

and

$$\left\|A'\left(x_1^0, \ldots, x_N^0\right)\right\| \geq \left\|FA\left(x_1^0, \ldots, x_N^0\right)\right\| \geq \left\|A\left(x_1^0, \ldots, x_N^0\right)\right\| - \|FA - A\| > 1 - \frac{1}{2}\eta\left(\frac{\varepsilon}{2}\right).$$

Since $\dim(A'(X_1 \times \ldots \times X_N)) < \infty$, there is $k \in \mathbb{N}$ such that for every $(x_1, \ldots, x_N) \in X_1 \times \ldots \times X_N$,

$$A'(x_1, \ldots, x_N) = \sum_{i=1}^{k} A_i(x_1, \ldots, x_N)y_i$$

for some $A_i \in \mathcal{K}(X_1, \ldots, X_N) \setminus \{0\}$ and $y_i \in B_Y$ for $i = 1, \ldots, k$. We set $M := \max\left\{\left\|x_i^0\right\| : j = 1, \ldots, N\right\}$ and we choose α such that

$$0 < \alpha < \min\left\{\frac{1}{4kM}\eta\left(\frac{\varepsilon}{2}\right), \frac{1}{8kM}\varepsilon\right\}.$$

By [29, Theorem 3.1], there are a natural number $m \in \mathbb{N}$ and a subspace E of Y which is linearly isometric to ℓ_∞^m such that $d(y, E) < \alpha$ for every $y \in B_Y \cap A'(X_1 \times \ldots \times X_N)$. In particular, for $i = 1, \ldots, k$, there is $e_i \in E$ such that $\|e_i - y_i\| < \alpha$. Define $C \in \mathcal{K}(X_1, \ldots, X_N; E)$ by

$$C(x_1, \ldots, x_N) := \sum_{i=1}^{k} A_i(x_1, \ldots, x_N)e_i \quad ((x_1, \ldots, x_N) \in X_1 \times \ldots \times X_N).$$

Then $\|C - A'\| < kM\alpha$. This implies that $0 < 1 - kM\alpha < \|C\| < 1 + kM\alpha$. Moreover,

$$\left\|C\left(x_1^0, \ldots, x_N^0\right)\right\| > \left\|A'\left(x_1^0, \ldots, x_N^0\right)\right\| - \|C - A'\| > 1 - \frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right) - kM\alpha.$$

and then

$$\left\|\left(\frac{C}{\|C\|}\right)\left(x_1^0, \ldots, x_N^0\right)\right\| > \frac{1 - \frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right) - kM\alpha}{1 + kM\alpha} > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Since E is isometric to ℓ_∞^m, $(X_1, \ldots, X_N; E)$ has the BPBp for compact multilinear mappings with the function η and so there are $B \in \mathcal{L}(X_1, \ldots, X_N; E) \subset \mathcal{K}(X_1, \ldots, X_N; Y)$ with $\|B\| = 1$ and $\left(z_1^0, \ldots, z_N^0\right) \in S_{X_1} \times \ldots \times S_{X_N}$ such that

$$\left\|B\left(z_1^0, \ldots, z_N^0\right)\right\| = 1, \quad \max_{1 \le j \le N} \|z_j^0 - x_j^0\| < \frac{\varepsilon}{2} < \varepsilon \quad \text{and} \quad \left\|B - \frac{C}{\|C\|}\right\| < \frac{\varepsilon}{2}.$$

It remains to prove that $\|B - A\| < \varepsilon$. This is true since

$$\|B - A\| \le \left\|B - \frac{C}{\|C\|}\right\| + \left\|\frac{C}{\|C\|} - C\right\| + \|C - A'\| + \|A' - A\| < \frac{\varepsilon}{2} + 2kM\alpha + \frac{\varepsilon}{4} < \varepsilon.$$

This shows that $(X_1, \ldots, X_N; Y)$ has the BPBp for compact multilinear mappings. ∎

From Proposition 2.3 and Theorem 2.9, for a predual Y of an L_1-space we see that $(X_1, \ldots, X_N; Y)$ has the BPBp for multilinear mappings whenever X_i is finite dimensional for each i. Moreover, it is known that $(C_0(L), C_0(K))$ has the BPBp for bilinear forms in the complex case [26, Theorem 2] and $(L_1(\mu), c_0)$ has the BPBp for bilinear forms [3, Corollary 2.7(2)]. Also, [5, Theorem 2.2] shows that (X_1, X_2) has the same property whenever X_1 and X_2 are uniformly convex spaces. Hence, we deduce the following corollary.

Corollary 2.10. *For a predual Y of an L_1-space, $(X, Z; Y)$ has the BPBp for compact bilinear mappings in the following cases. (a) Complex Banach spaces $X = C_0(L)$ and $Z = C_0(K)$ where L and K are locally compact topological Hausdorff spaces. (b) $X = L_1(\mu)$ and $Z = c_0$. (c) X and Z uniformly convex Banach spaces.*

Adapting Theorem 2.9 we can prove the analogous result for compact symmetric multilinear mappings and for compact N-homogeneous polynomials.

3. The generalized AHSP for bilinear forms

In this section we study the *generalized approximate hyperplane series property for bilinear forms* which is motivated by the AHSP and the generalized AHSP (see [1] and [28]). The AHSP appears for the first time in [1] where the authors were interested in characterizing the pair $(\ell_1; Y)$ to have the BPBp for operators. They showed that the pair $(\ell_1; Y)$ has the BPBp for operators if and only if the Banach space Y has the AHSP [1, Theorem 4.1]. Similarly, Kim, Lee and Martín defined the generalized AHSP (see [28, Definition 4]) and they use it to characterize the pair $(\ell_1(X); Y)$ to have the BPBp for operators. In this section, we get the analogous result for bilinear forms.

Definition 3.1. Let X and Y be Banach spaces. We say that the pair (X, Y) has the *generalized approximate hyperplane series property for bilinear forms* (generalized AHSP for bilinear forms, for short) if for every $\varepsilon > 0$, there is $0 < \eta(\varepsilon) < \varepsilon$ such that for given sequences $(T_k)_k \subset \mathcal{L}(X; Y^*)$ with $\|T_k\| = 1$ for every $k \in \mathbb{N}$ and $(x_k)_k \subset S_X$, an element $y_0 \in S_Y$ and a convex series $\sum_{k=1}^\infty \alpha_k$ with

$$\operatorname{Re} \sum_{k=1}^{\infty} \alpha_k T_k(x_k)(y_0) > 1 - \eta(\varepsilon),$$

there are $u_0 \in S_Y$, a subset $A \subset \mathbb{N}$ and sequences $(S_k)_k \subset \mathcal{L}(X; Y^*)$ with $\|S_k\| = 1$ for every $k \in \mathbb{N}$ and $(z_k)_k \subset S_X$ satisfying

(1) $\sum_{k \in A} \alpha_k > 1 - \varepsilon$,
(2) $\|z_k - x_k\| < \varepsilon$ and $\|S_k - T_k\| < \varepsilon$ for all $k \in A$,
(3) $\|u_0 - y_0\| < \varepsilon$ and
(4) $S_k(z_k)(u_0) = 1$ for all $k \in A$.

Although there is no bilinear forms in the above definition, we put its name as *AHSP for bilinear forms* since it implies the BPBp for bilinear forms to the pair (X, Y). As the first result in this section, we observe that the pair (X, Y) has the generalized AHSP for bilinear forms whenever X and Y are finite dimensional Banach spaces. To do so, we will use the following technical lemma which is proved in [1].

Lemma 3.2. *Let $\{c_n\}$ be a sequence of scalars with $|c_n| \leq 1$ for every $n \in \mathbb{N}$ and let $\sum_{n=1}^{\infty} \alpha_n$ be a convex series such that* $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$ *for some $\eta > 0$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$ satisfies the estimate* $\displaystyle\sum_{i \in A} \alpha_i \geq 1 - \frac{\eta}{1-r}$.

Proposition 3.3. *For every finite dimensional Banach spaces X and Y, the pair (X, Y) has the generalized AHSP for bilinear forms.*

Proof. We first claim that for arbitrary $\varepsilon > 0$, there exists a positive real number $\eta(\varepsilon) > 0$ satisfying the following. For each $y_0 \in S_Y$, there is $u_0 \in S_Y$ with $\|u_0 - y_0\| < \varepsilon$ such that whenever $(x, T) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ satisfies $\operatorname{Re} T(x)(y_0) > 1 - \eta(\varepsilon)$, there exists $(z, S) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ with $S(z)(u_0) = 1$ and $\|S - T\|, \|z - x\| < \varepsilon$. Otherwise, there are $\varepsilon_0 > 0$ and $(y_n)_{n\in\mathbb{N}} \subset S_Y$ such that for each $u \in S_Y$ with $\|u - y_n\| < \varepsilon_0$, there exists $(x_n^u, T_n^u) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ with $\operatorname{Re} T_n^u(x_n^u)(y_n) > 1 - \frac{1}{n}$ satisfying that if $(z, S) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ satisfies $S(z)(u) = 1$ then $\max\{\|S - T_n^u\|, \|z - x_n^u\|\} \geq \varepsilon_0$. Since Y is finite dimensional, we assume that y_n conveges to $y_\infty \in S_Y$ and $\|y_n - y_\infty\| < \varepsilon$ for each n. Using compactness again, we may assume that $(x_n^{y_\infty}, T_n^{y_\infty})$ converges to $(x_\infty, T_\infty) \in S_X \times S_{\mathcal{L}(X;Y^*)}$. Then $T_\infty(x_\infty)(y_\infty) = 1$ and this gives a contradiction since we would have that $\max\{\|T_n^{y_\infty} - T_\infty\|, \|x_n^{y_\infty} - x_\infty\|\} \geq \varepsilon_0$.

Consider the sequences $(T_k)_k \subset S_{\mathcal{L}(X;Y^*)}$ and $(x_k)_k \subset S_X$, a convex series $\sum_{k=1}^{\infty} \alpha_k$ and an element $y_0 \in S_Y$ satisfying that $\operatorname{Re} \sum_{k=1}^{\infty} \alpha_k T_k(x_k)(y_0) > 1 - (\eta(\varepsilon))^2$. By Lemma 3.2, we have $\sum_{k\in A} \alpha_k > 1 - \varepsilon$ where $A := \{k \in \mathbb{N} : \operatorname{Re} T_k(x_k)(y_0) > 1 - \eta(\varepsilon)\}$. Also, there are $u_0 \in S_Y$ and $(z_k, S_k)_{k\in A} \subset S_X \times S_{\mathcal{L}(X;Y^*)}$ such that $\|u_0 - y_0\| < \varepsilon$, $S_k(z_k)(u_0) = 1$, $\|S_k - T_k\| < \varepsilon$ and $\|z_k - x_k\| < \varepsilon$ for all $k \in A$. This shows that the pair (X, Y) has the generalized AHSP for bilinear forms. ∎

As we mentioned before it is easy to see that if the pair (X, Y) has the generalized AHSP for bilinear forms, then it has the BPBp for bilinear forms. In the following proposition we prove that the converse holds when Y is a Hilbert space.

Proposition 3.4. *Let X be a Banach space and let H be a Hilbert space. The pair (X, H) has the BPBp for bilinear forms if and only if the pair (X, H) has the generalized AHSP.*

Proof. Assume that the pair (X, H) has the BPBp for bilinear forms with function $\eta(\cdot)$. Note that since H is a Hilbert space, there exists a function $\xi(\cdot) > 0$ satisfying that $\lim_{t\to 0} \xi(t) = 0$ and that for every $\varepsilon > 0$ and points $h_1, h_2 \in S_H$ with $\|h_1 - h_2\| < \varepsilon$, there exits a linear isometry $R : H \longrightarrow H$ such that $R(h_1) = h_2$ and $\|R - Id_H\| < \xi(\varepsilon)$.

Fix $\varepsilon > 0$ and choose $\varepsilon' > 0$ so that $\sqrt{2(\eta(\varepsilon') + 3\varepsilon)} + \varepsilon' + \xi(\varepsilon') < \varepsilon$. Consider a sequences $(T_k)_k \subset S_{\mathcal{L}(X;H^*)}$ and $(x_k)_k \subset S_X$, an element $h_0 \in S_H$ and a convex series $\sum_{n=1}^{\infty} \alpha_n$ such that $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_k T_k(x_k)(h_0) > 1 - (\eta(\varepsilon'))^2$. By Lemma 3.2 we get that $\sum_{k\in A} \alpha_k > 1 - \eta(\varepsilon') > 1 - \varepsilon'$ where $A := \{k \in \mathbb{N} : \operatorname{Re} T_k(x_k)(y_0) > 1 - \eta(\varepsilon')\}$. For each $k \in A$, we define a bilinear form B_k on $X \times H$ by $B_k(x, h) := T_k(x)(h)$ for all $(x, h) \in X \times H$. Then $\|B_k\| = \|T_k\| = 1$ for all $k \in A$ and $\operatorname{Re} B_k(x_k, h_0) = \operatorname{Re} T_k(x_k)(h_0) > 1 - \eta(\varepsilon')$, for every $k \in A$. From the assumption, there are a bilinear form C_k and $(z_k, u_k) \in S_X \times S_H$ such that

$$|C_k(z_k, u_k)| = 1 = \|C_k\|, \quad \|z_k - x_k\| < \varepsilon', \quad \|u_k - h_0\| < \varepsilon' \quad \text{and} \quad \|C_k - B_k\| < \varepsilon'$$

for all $k \in A$.

Choose a scalar c_k with $|c| = 1$, such that $c_k C_k(z_k, u_k) = 1$. Then,

$$\begin{aligned}|1 - c_k| &\leq \sqrt{2(1 - \text{Re } c_k)} = \sqrt{2(1 - \text{Re } C_k(z_k, u_k))} \\ &\leq \sqrt{2\left(|1 - \text{Re } B_k(x_k, h_0)| + |\text{Re } B_k(x_k, h_0) - \text{Re } B_k(z_k, u_k)| + |\text{Re } B_k(z_k, u_k) - \text{Re } C_k(z_k, u_k)|\right)} \\ &< \sqrt{2(\eta(\varepsilon') + 3\varepsilon')}\end{aligned}$$

For each $k \in A$, there exits a linear isometry $R_k : H \longrightarrow H$ such that $R_k(h_0) = u_k$ and $\|R_k - Id_H\| < \xi(\varepsilon')$. Define $S_k \in \mathcal{L}(X; H^*)$ by $S_k(x)(h) := c_k C_k(x, R_k(h))$ for every $x \in X$ and $h \in H$. Then $\|S_k - T_k\| < \sqrt{2(\eta(\varepsilon') + 3\varepsilon')} + \varepsilon' + \xi(\varepsilon') < \varepsilon$ and $\|S_k\| = S_k(x_k)(h_0) = 1$ for every $k \in A$. ∎

We now show that the generalized AHSP for bilinear forms characterizes the pair $(\ell_1(X), Y)$ to have the BPBp for bilinear forms. This is the analogous version of [28, Theorem 6].

Theorem 3.5. *Let X and Y be Banach spaces. The pair (X, Y) has the generalized AHSP for bilinear forms if and only if the pair $(\ell_1(X), Y)$ has the BPBp for bilinear forms.*

Proof. The proof will be given for complex Banach spaces since the real case is not only similar but also simpler. Suppose that the pair (X, Y) has the generalized AHSP for bilinear forms with $0 < \eta(\varepsilon) < \varepsilon$ for $\varepsilon \in (0, 1)$.

Let B be a bilinear form defined on $\ell_1(X) \times Y$ with $\|B\| = 1$ and $(x_0, y_0) \in S_{\ell_1(X)} \times S_Y$ satisfying $|B(x_0, y_0)| > 1 - \eta(\varepsilon/3)$. Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ be such that $\alpha B(x_0, y_0) = B(x_0, \alpha y_0) = |B(x_0, y_0)|$. Define $T : \ell_1(X) \longrightarrow Y^*$ by $T(x)(y) := B(x, y)$ for all $x \in \ell_1(X)$ and $y \in Y$. Then $\|T\| = \|B\| = 1$. We denote by T_k the restriction of T on the k-th coordinate X of $\ell_1(X)$. Then, we see that $T(x) = \sum_{k \in \mathbb{N}} T_k(x_k)$ for every $x = (x_k)_k \in \ell_1(X)$. Since $x_0 \in S_{\ell_1(X)}$, we may write $x_0 = \left(\alpha_k x_k^0\right)_k$ with $\sum_{k=1}^{\infty} \alpha_k = 1$, $\alpha_k \geq 0$ and $x_k^0 \in S_X$ for all $k \in \mathbb{N}$. Then

$$\sum_{k=1}^{\infty} \alpha_k T_k(x_k^0)(\alpha y_0) = \alpha \sum_{k=1}^{\infty} T_k(\alpha_k x_k^0)(y_0) = \alpha T(x_0)(y_0) = \alpha B(x_0, y_0) = |B(x_0, y_0)| > 1 - \eta\left(\frac{\varepsilon}{3}\right).$$

Then there are $u_0 \in S_Y$, a set $A \subset \mathbb{N}$ and sequences $(S_k)_k \subset S_{\mathcal{L}(X, Y^*)}$ and $(z_k)_k \subset S_X$ such that

(1) $\sum_{k \in A} \alpha_k > 1 - \frac{\varepsilon}{3}$,
(2) $\left\|z_k - x_k^0\right\| < \frac{\varepsilon}{3}$ and $\|S_k - T_k\| < \frac{\varepsilon}{3}$ for all $k \in A$,
(3) $\|u_0 - \alpha y_0\| < \frac{\varepsilon}{3} < \varepsilon$ and
(4) $S_k(z_k)(u_0) = 1$ for all $k \in A$.

Define $S : \ell_1(X) \longrightarrow Y^*$ by $S(x) := \sum_{k \in A} S_k(x_k) + \sum_{k \in \mathbb{N} \setminus A} T_k(x_k)$ for $x = (x_k)_k \in \ell_1(X)$. Then we get that $\|S\| = 1$. Define now a bilinear form C on $\ell_1(X) \times Y$ by $C(x, y) := S(x)(y)$ for all $(x, y) \in \ell_1(X) \times Y$. So $\|C\| = \|S\| = 1$ and $\|C - B\| < \varepsilon$. Let $\beta_k = \frac{\alpha_k}{\sum_{k \in A} \alpha_k}$ for $k \in A$ and $\beta_k = 0$ otherwise. We define $z_0 := (\beta_k z_k)_k \in \ell_1(X)$ where $z_k = x_k^0$ for all $k \in \mathbb{N} \setminus A$. Then $\|z_0\|_1 = \sum_{k \in A} \beta_k = 1$ and

$$\begin{aligned}\|z_0 - x_0\|_1 &= \sum_{k \in A} \left\|\beta_k z_k - \alpha_k x_k^0\right\| + \sum_{k \in \mathbb{N} \setminus A} \|\alpha_k x_k^0\| \\ &= \sum_{k \in A} \left\| \frac{\alpha_k}{\sum_{j \in A} \alpha_j} z_k - \alpha_k z_k \right\| + \sum_{k \in A} \|\alpha_k z_k - \alpha_k x_k^0\| + \sum_{k \in \mathbb{N} \setminus A} \alpha_k \\ &\overset{(1),(2)}{<} 1 - \sum_{k \in A} \alpha_k + \frac{\varepsilon}{3} \sum_{k \in A} \alpha_k + \frac{\varepsilon}{3} \overset{(1)}{<} \varepsilon.\end{aligned}$$

Also, by using (3), we get that $\|\alpha^{-1} u_0 - y_0\|_1 = \|u_0 - \alpha y_0\|_1 < \varepsilon$. Finally,

$$1 \geq |C(z_0, \alpha^{-1} u_0)| = |S(z_0)(u_0)| = \left|\sum_{k \in A} S_k(\beta_k z_k)(u_0)\right| = \left|\sum_{k \in A} \beta_k S_k(z_k)(u_0)\right| = \sum_{k \in A} \beta_k = 1.$$

Now we assume that the pair $(\ell_1(X), Y)$ has the BPBp for bilinear forms with function $\eta(\varepsilon) < \varepsilon$ for $\varepsilon > 0$. Let $\xi(\varepsilon) > 0$ be such that

$$\xi(\varepsilon) + \frac{2\xi(\varepsilon)}{\varepsilon} < \varepsilon \quad \text{and} \quad \xi(\varepsilon) + \sqrt{2(\eta(\xi(\varepsilon)) + 3\xi(\varepsilon))} < \varepsilon. \tag{3.1}$$

Consider sequences $(T_k)_k \subset S_{\mathcal{L}(X;Y^*)}$ and $\left(x_k^0\right)_k \subset S_X$, a convex series $\sum_{k=1}^\infty \alpha_k$ and $y_0 \in S_Y$ such that

$$\text{Re} \sum_{k=1}^\infty \alpha_k T_k(x_k^0)(y_0) > 1 - \eta(\xi(\varepsilon)).$$

We define a norm 1 bilinear form B on $\ell_1(X) \times Y$ by

$$B(x,y) := \sum_{k=1}^\infty T_k(x_k)(y) \quad (((x_k)_k, y) = (x,y) \in \ell_1(X) \times Y).$$

Put $x_0 := \left(\alpha_k x_k^0\right) \in S_{\ell_1(X)}$. Then $\text{Re } B(x_0, y_0) = \text{Re } \sum_{k=1}^\infty T_k\left(\alpha_k x_k^0\right)(y_0) > 1 - \eta(\xi(\varepsilon))$. Since $(\ell_1(X), Y)$ has the BPBp for bilinear forms, there are a bilinear form C on $\ell_1(X) \times Y$ and $(z_0, u_0) \in S_{\ell_1(X)} \times S_Y$ such that

$$|C(z_0, u_0)| = 1 = \|C\|, \quad \|z_0 - x_0\| < \varepsilon, \quad \|u_0 - y_0\| < \varepsilon \text{ and } \|C - B\| < \xi(\varepsilon) < \varepsilon.$$

Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ satisfy that $\alpha C(z_0, u_0) = C(z_0, \alpha u_0) = 1$. Note that we have

$$\text{Re } C(z_0, u_0) \geq \text{Re } B(x_0, y_0) - \|C - B\| - \|z_0 - x_0\| - \|u_0 - y_0\| > 1 - \eta(\xi(\varepsilon)) - 3\xi(\varepsilon).$$

Since $|\alpha| = 1$, we see that $|1 - \alpha| = \sqrt{2(1 - \text{Re } \alpha)} = \sqrt{2(1 - \text{Re } C(z_0, u_0))}$. From the last inequality, we conclude that

$$\|y_0 - \alpha u_0\| \leq \|y_0 - u_0\| + |1 - \alpha| < \xi(\varepsilon) + \sqrt{2(\eta(\xi(\varepsilon)) + 3\xi(\varepsilon))} \overset{(3.1)}{<} \varepsilon.$$

To find the desired set A, we write $z_0 = (z_k^0)_k \in S_{\ell_1(X)}$ and consider a set $\widetilde{A} := \left\{k \in \mathbb{N} : \|z_k^0\| = 0\right\}$. Note that

$$\xi(\varepsilon) > \|x_0 - z_0\|_1 = \sum_{k \in \mathbb{N}} \|\alpha_k x_k^0 - z_k^0\| \geq \sum_{k \in \widetilde{A}} \alpha_k \tag{3.2}$$

and also that

$$\begin{aligned} \|x_0 - z_0\|_1 &\geq \sum_{k \in \mathbb{N} \setminus \widetilde{A}} \|\alpha_k x_k^0 - z_k^0\| = \sum_{k \in \mathbb{N} \setminus \widetilde{A}} \alpha_k \left\| x_k^0 - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|} \right\| \\ &\geq \sum_{k \in \mathbb{N} \setminus \widetilde{A}} \left(\alpha_k \left\| x_k^0 - \frac{z_k^0}{\|z_k^0\|} \right\| - \left\| \frac{z_k^0}{\|z_k^0\|} - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|} \right\| \right). \end{aligned}$$

Observe that for $k \in \mathbb{N} \setminus \widetilde{A}$

$$\left\| \frac{z_k^0}{\|z_k^0\|} - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|} \right\| = \left| 1 - \frac{\|z_k^0\|}{\alpha_k} \right| = \left| \|x_k^0\| - \frac{\|z_k^0\|}{\alpha_k} \cdot \left\| \frac{z_k^0}{\|z_k^0\|} \right\| \right| \leq \left\| x_k^0 - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|} \right\|.$$

So

$$\begin{aligned} \sum_{k \in \mathbb{N} \setminus \widetilde{A}} \alpha_k \left\| x_k^0 - \frac{z_k^0}{\|z_k^0\|} \right\| &\leq \|x_0 - z_0\|_1 + \sum_{k \in \mathbb{N} \setminus \widetilde{A}} \alpha_k \left\| x_k^0 - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|} \right\| \\ &= \|x_0 - z_0\|_1 + \sum_{k \in \mathbb{N} \setminus \widetilde{A}} \|\alpha_k x_k^0 - z_k^0\| \leq 2\|x_0 - z_0\|_1 < 2\xi(\varepsilon). \end{aligned}$$

Define $A := \left\{k \in \mathbb{N} : \left\| x_k^0 - \frac{z_k^0}{\|z_k^0\|} \right\| < \varepsilon, \ \|z_k^0\| \neq 0 \right\}$. We see that

$$2\xi(\varepsilon) > \sum_{k \in \mathbb{N} \setminus \widetilde{A}} \alpha_k \left\| x_k^0 - \frac{z_k^0}{\|z_k^0\|} \right\| \geq \sum_{k \in (\mathbb{N} \setminus \widetilde{A}) \setminus A} \alpha_k \left\| x_k^0 - \frac{z_k^0}{\|z_k^0\|} \right\| \geq \varepsilon \sum_{k \in (\mathbb{N} \setminus \widetilde{A}) \setminus A} \alpha_k. \tag{3.3}$$

Since $1 = \sum_{k \in \mathbb{N}} \alpha_k = \sum_{k \in A} \alpha_k + \sum_{k \in (\mathbb{N} \setminus \widetilde{A}) \setminus A} \alpha_k + \sum_{k \in \widetilde{A}} \alpha_k$, we get that

$$\sum_{k \in A} \alpha_k = 1 - \sum_{k \in \widetilde{A}} \alpha_k - \sum_{k \in (\mathbb{N} \setminus \widetilde{A}) \setminus A} \alpha_k \overset{(3.2)}{>} 1 - \xi(\varepsilon) - \sum_{k \in (\mathbb{N} \setminus \widetilde{A}) \setminus A} \alpha_k \overset{(3.3)}{>} 1 - \xi(\varepsilon) - \frac{2\xi(\varepsilon)}{\varepsilon} \overset{(3.1)}{>} 1 - \varepsilon.$$

Now we define $S : \ell_1(X) \longrightarrow Y^*$ by $S(x)(y) := C(x,y)$ for all $x \in \ell_1(X)$ and $y \in Y$ and let S_k be the restriction of S on the k-th coordinate X of $\ell_1(X)$. Then, we have that $\|S_k - T_k\| < \|C - B\| < \varepsilon$ and

$$S(z_0)(\alpha u_0) = C(z_0, \alpha u_0) = 1.$$

Since $1 = \|z_0\|_1 = \sum_{k=1}^{\infty} \|z_k^0\|$, we get that $S_k(z_k^0)(\alpha u_0) = \|z_k^0\|$ for all $k \in \mathbb{N}$. Therefore, for every $k \in A$, we see that $S_k\left(\frac{z_k^0}{\|z_k^0\|}\right)(\alpha u_0) = 1$ which implies $\|S_k\| = 1$. The proof ends if we choose the set A, the sequences $(S_k)_k \subset \mathcal{L}(X; Y^*)$ which satisfies $\|S_k\| = 1$ for all $k \in \mathbb{N}$ and $(z_k)_k := \left(\frac{z_k^0}{\|z_k^0\|}\right)_k \subset S_X$ and the element $\alpha u_0 \in S_Y$. ∎

4. The numerical radius on $\mathcal{L}(^N L_1(\mu); L_1(\mu))$

In this section we work with numerical radius of an N-linear mapping. We consider a set

$$\Pi_N(X) := \{(x_1, \ldots, x_N, x^*) \in S_X \times \ldots \times S_X \times S_{X^*} : x^*(x_1) = \ldots = x^*(x_N) = 1\}.$$

When $N = 1$, we denote $\Pi_1(X)$ by $\Pi(X)$. The numerical radius of an N-linear mapping $A \in \mathcal{L}^N(X; X)$ is defined by

$$v(A) := \sup\{|x^*(A(x_1, \ldots, x_N))| : (x_1, \ldots, x_N, x^*) \in \Pi_N(X)\}.$$

As in the operator case we have that v is a semi-norm on $\mathcal{L}(^N X; X)$ such that $v(A) \leq \|A\|$ for all $A \in \mathcal{L}(^N X; X)$. On the other hand, the equality is not true in general. Nevertheless, in [17, Theorem 3.1(i) and Theorem 3.2] it was proved that $v(A) = \|A\|$ for every $A \in \mathcal{L}(^N c_0; c_0)$ or $A \in \mathcal{L}(^N \ell_1; \ell_1)$. Also in [16, Theorem 3.2] it was proved that $v(L) = \|L\|$ for every $L \in \mathcal{L}(^N A_D; A_D)$ where A_D is the disc algebra. We start this section by proving that the same happens in an $L_1(\mu)$-space for an arbitrary measure μ. Before we do that, we provide a technical lemma relating the numerical radius of a multilinear mapping and direct sums.

Given a family $\{X_\lambda\}_{\lambda \in \Lambda}$ of Banach spaces, we denote by $\left[\bigoplus_{\lambda \in \Lambda} X_\lambda\right]_{c_0}$ and $\left[\bigoplus_{\lambda \in \Lambda} X_\lambda\right]_{\ell_1}$ the c_0 and ℓ_1-sum of $\{X_\lambda\}$. Letting $X = \left[\bigoplus_{\lambda \in \Lambda} X_\lambda\right]_{c_0}$ or $\left[\bigoplus_{\lambda \in \Lambda} X_\lambda\right]_{\ell_1}$, we consider $X^* = \left[\bigoplus_{\lambda \in \Lambda} X_\lambda^*\right]_{\ell_1}$ or $\left[\bigoplus_{\lambda \in \Lambda} X_\lambda^*\right]_{\ell_\infty}$ to be their dual spaces, respectively. We denote by $P_\lambda : X \longrightarrow X_\lambda$ the norm-one linear projection from X onto X_λ and by $Q_\lambda : X^* \longrightarrow X_\lambda^*$ the norm-one linear projection from X^* onto X_λ^*. We denote an element $x \in \bigoplus_{\lambda \in \Lambda} X_\lambda$ by $x = (x_\lambda)_{\lambda \in \Lambda}$.

In the next theorem, we prove that the norm and the numerical radius of a multilinear mapping defined on $L_1(\mu) \times \ldots \times L_1(\mu)$ coincide. In what follows, if $T \in \mathcal{L}(X; Y)$, then $T^* \in \mathcal{L}(Y^*; X^*)$ denotes its adjoint and $\langle x, x^* \rangle$ means the action $x^*(x)$ for $x \in X$ and $x^* \in X^*$.

Theorem 4.1. *Let (Ω, Σ, μ) be a measure space. For each positive integer N and each $A \in \mathcal{L}(^N L_1(\mu); L_1(\mu))$, we have $v(A) = \|A\|$.*

Proof. Since $v(A) \leq \|A\|$, we need to prove the another inequality. Without loss of generality, suppose that $\|A\| = 1$. Let $\varepsilon \in (0, 1)$ and choose $f_1, \ldots, f_N \in S_{L_1(\mu)}$ such that

$$\|A(f_1, \ldots, f_N)\|_1 > 1 - \frac{\varepsilon}{2}. \tag{4.1}$$

We prove that $v(A) > 1 - \varepsilon$. Consider

$$\Omega' = \left(\cup_{i=1}^N \{t \in \Omega \ : \ |f_i(t)| > 0\}\right) \cup \{t \in \Omega \ : \ |A(f_1, \ldots, f_N)(t)| > 0\}.$$

For a partition $\pi \subset \Sigma$ of Ω' into a countable family of disjoint measurable sets with positive measure, define a projection $E_\pi : L_1(\mu) \longrightarrow L_1(\mu)$ given by for each $f \in L_1(\mu)$

$$E_\pi(f) := \sum_{F \in \pi} \left(\frac{1}{\mu(F)} \int_F f d\mu\right) \chi_F.$$

Since Ω' is a countable union of meaurable subsets having finite measure, we apply [21, Lemma III.2.1, pg. 67] to the finite measurable subsets of Ω', f_i and $A(f_1, \ldots, f_N)$ to find a partition π_0 such that for every $\pi \geq \pi_0$

$$\|E_\pi f_j - f_j\|_1 < \frac{\varepsilon}{2(N+1)} \tag{4.2}$$

and that

$$\|E_\pi A(f_1, \ldots, f_N) - A(f_1, \ldots, f_N)\|_1 < \frac{\varepsilon}{2(N+1)}. \tag{4.3}$$

We claim that for all $\pi \geq \pi_0$, we have the following inequality

$$\|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N)\|_1 > 1 - \varepsilon. \tag{4.4}$$

Indeed, note first that since A is N-linear mapping, we have that

$$A(E_\pi f_1, \ldots E_\pi f_N) - A(f_1, \ldots, f_N) = A(E_\pi f_1 - f_1, \ldots, E_\pi f_N) + A(f_1, E_\pi f_2 - f_2, \ldots, E_\pi f_N) + \ldots + A(f_1, \ldots, f_{n-1}, E_\pi f_n - f_n),$$

and then

$$\|A(E_\pi f_1, \ldots E_\pi f_N) - A(f_1, \ldots, f_N)\|_1 \leq \sum_{j=1}^{N} \|E_\pi f_j - f_j\|_1 < N \cdot \frac{\varepsilon}{2(N+1)}$$

This shows that since $\|E_\pi\| \leq 1$,

$$\|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N) - E_\pi A(f_1, \ldots, f_N)\|_1 < N \cdot \frac{\varepsilon}{2(N+1)}.$$

On the other hand, by using (4.3) and (4.1), we get for all $\pi \geq \pi_0$ that

$$\|E_\pi A(f_1, \ldots, f_N)\|_1 > \|A(f_1, \ldots, f_N)\|_1 - \frac{\varepsilon}{2(N+1)} > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2(N+1)}.$$

So

$$\begin{aligned} \|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N)\|_1 &\geq \|E_\pi A(f_1, \ldots, f_N)\|_1 - \|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N) - E_\pi A(f_1, \ldots, f_N)\|_1 \\ &> 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2(N+1)} - N \cdot \frac{\varepsilon}{2(N+1)} = 1 - \varepsilon. \end{aligned}$$

Now we fix $\pi_0 = \{F_i \,:\, i \in \mathbb{N}\} \subset \Sigma$. For each $j = 1, \ldots, N$, we put

$$E_{\pi_0} f_j = \sum_i a_i^j \cdot \frac{1}{\mu(F_i)} \chi_{F_i} \quad \text{with} \quad a_i^j = \int_{F_i} f_j d\mu.$$

By using (4.2),

$$\begin{aligned} 1 - \frac{\varepsilon}{2(N+1)} < \|E_{\pi_0} f_j\|_1 = \int_\Omega |E_{\pi_0} f_j| d\mu &= \int_\Omega \left| \sum_i a_i^j \frac{1}{\mu(F_i)} \chi_{F_i} \right| d\mu \\ &= \int_\Omega \sum_i |a_i^j| \frac{1}{\mu(F_i)} \chi_{F_i} d\mu \\ &= \sum_i |a_i^j| \leq 1. \end{aligned}$$

Hence, by using (4.4), we have that

$$\begin{aligned} 1 - \varepsilon &< \|E_{\pi_0} A(E_{\pi_0} f_1, \ldots, E_{\pi_0} f_N)\|_1 \\ &= \left\| E_{\pi_0} A \left(\sum_i a_i^1 \frac{1}{\mu(F_i)} \chi_{F_i}, \ldots, \sum_i a_i^N \frac{1}{\mu(F_i)} \chi_{F_i} \right) \right\|_1 \\ &= \left\| \sum_{l_1, \ldots, l_N \in \mathbb{N}} a_{l_1}^1 \cdots a_{l_N}^N E_{\pi_0} A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) \right\|_1 \\ &\leq \sum_{l_1, \ldots, l_N \in \mathbb{N}} |a_{l_1}^1| \cdots |a_{l_N}^N| \left\| E_{\pi_0} A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) \right\|_1. \end{aligned}$$

and so we conclude that there exist $l_1, \ldots, l_N \in \mathbb{N}$ such that

$$\left\| E_{\pi_0} A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) \right\|_1 > 1 - \varepsilon. \tag{4.5}$$

Now we write

$$E_{\pi_0} A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) = \sum_i a_i \frac{1}{\mu(F_i)} \chi_{F_i},$$

where

$$a_i = \int_{F_i} A\left(\frac{1}{\mu(F_{l_1})}\chi_{F_{l_1}}, \dots, \frac{1}{\mu(F_{l_N})}\chi_{F_{l_N}}\right) d\mu.$$

Define on $L_\infty(\mu)$ the element

$$g := \sum_i c_i \chi_{F_i}, \text{ where } |c_i| = 1, \ c_i a_i = |a_i| \text{ for all } i \in \mathbb{N}.$$

Then $\|g\|_\infty = 1$, we note that for all $j = 1, \dots, N$, we have that

$$\left\langle \frac{\overline{c_{l_j}}}{\mu(F_{l_j})}\chi_{F_{l_j}}, g \right\rangle = \int_\Omega \frac{\overline{c_{l_j}}}{\mu(F_{l_j})}\chi_{F_{l_j}} \cdot g(t) d\mu(t) = \frac{1}{\mu(F_{l_j})}\int_\Omega [\chi_{F_{l_j}}(t)]^2 d\mu(t) = 1$$

and also

$$\begin{aligned}
\left|\left\langle g, E_{\pi_0} A\left(\frac{\overline{c_{l_1}}}{\mu(F_{l_1})}\chi_{F_{l_1}}, \dots, \frac{\overline{c_{l_N}}}{\mu(F_{l_N})}\chi_{F_{l_N}}\right)\right\rangle\right| &= \left|\left\langle g, E_{\pi_0} A\left(\frac{1}{\mu(F_{l_1})}\chi_{F_{l_1}}, \dots, \frac{1}{\mu(F_{l_N})}\chi_{F_{l_N}}\right)\right\rangle\right| \\
&= \left|\int_\Omega \left(\sum_i c_i \chi_{F_i}\right) \cdot \left(\sum_i a_i \frac{1}{\mu(F_i)}\chi_{F_i} d\right) \mu\right| \\
&= \sum_i |a_i| \\
&= \left\|E_{\pi_0} A\left(\frac{1}{\mu(F_{l_1})}\chi_{F_{l_1}}, \dots, \frac{1}{\mu(F_{l_N})}\chi_{F_{l_N}}\right)\right\|_1 \\
&\overset{(4.5)}{>} 1 - \varepsilon.
\end{aligned}$$

Finally, we consider the adjoint operator $E^*_{\pi_0} : L_\infty(\mu) \longrightarrow L_\infty(\mu)$. Then $\|E^*_{\pi_0}(g)\|_\infty \leq 1$. Also for all $j = 1, \dots, N$,

$$\left\langle \frac{\overline{c_{l_j}}}{\mu(F_{l_j})}\chi_{F_{l_j}}, E^*_{\pi_0}(g) \right\rangle = \left\langle E_{\pi_0}\left(\frac{\overline{c_{l_j}}}{\mu(F_{l_j})}\chi_{F_{l_j}}\right), g \right\rangle = \left\langle \frac{\overline{c_{l_j}}}{\mu(F_{l_j})}\chi_{F_{l_j}}, g \right\rangle = 1$$

and

$$\left|\left\langle A\left(\frac{\overline{c_{l_1}}}{\mu(F_{l_1})}\chi_{F_{l_1}}, \dots, \frac{\overline{c_{l_N}}}{\mu(F_{l_N})}\chi_{F_{l_N}}, E^*_{\pi_0}(g)\right)\right\rangle\right| = \left|\left\langle E_{\pi_0} A\left(\frac{\overline{c_{l_1}}}{\mu(F_{l_1})}\chi_{F_{l_1}}, \dots, \frac{\overline{c_{l_N}}}{\mu(F_{l_N})}\chi_{F_{l_N}}\right), g \right\rangle\right| > 1 - \varepsilon.$$

Then $v(A) > 1 - \varepsilon$. This completes the proof. ∎

5. The BPBp-nu for multilinear mappings

In the section we study the Bishop-Phelps-Bollobás property for numerical radius in the multilinear case. Before we give the positive and negative results about it, we define this property as follows.

Definition 5.1. We say that a Banach space X has the *Bishop-Phelps-Bollobás property for numerical radius for multilinear mappings* (BPBp-nu for multilinear mappings, for short) if for every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $A \in \mathcal{L}(^N X; X)$ with $v(A) = 1$ and $(x^0_1, \dots, x^0_N, x^*_0) \in \Pi_N(X)$ satisfy

$$|x^*_0\left(A\left(x^0_1, \dots, x^0_N\right)\right)| > 1 - \eta(\varepsilon),$$

there are $B \in \mathcal{L}(^N X; X)$ with $v(B) = 1$ and $(z^0_1, \dots, z^0_N, z^*_0) \in \Pi_N(X)$ such that

$$\left|z^*_0\left(B\left(z^0_1, \dots, z^0_N\right)\right)\right| = 1, \quad \max_{1 \leq j \leq N} \|z^0_j - x^0_j\| < \varepsilon, \quad \|z^*_0 - x^*_0\| < \varepsilon \quad \text{and} \quad \|B - A\| < \varepsilon.$$

If $N = 1$, then we go back to the operator case which we already commented some positive results in the Introduction. By following the step-by-step of [27, Proposition 2], we have that the finite-dimensional Banach spaces satisfy the BPBp-nu for multilinear mappings.

Proposition 5.2. *Let X be a finite-dimensional Banach space. Then X has the BPBp-nu for multilinear mappings.*

In the next theorem we prove that the infinite dimensional Banach space $L_1(\mu)$ fails the BPBp-nu for bilinear mappings although $L_1(\nu)$ has it in the operator case for every measure ν ([27, Theorem 4.1]; see also [23, Theorem 9]).

Theorem 5.3. *The infinite dimensional Banach space $L_1(\mu)$ does not satisfy the BPBp-nu for bilinear mappings.*

Proof. The proof is by contradiction. Suppose that $L_1(\mu)$ has the BPBp-nu for bilinear mappings with $\eta(\varepsilon)$ for a given $\varepsilon \in (0,1)$. Since $L_1(\mu)$ is infinite dimensional, we may consider measurable subsets $(E_k) \subset \Sigma$ such that $0 < \mu(E_k) < \infty$ and $E_i \cap E_j = \emptyset$ when $i \neq j$. We define $E_0 := \Omega \setminus \bigcup_{k=1}^{\infty} E_k \in \Sigma$ and a bilinear mapping $A : L_1(\mu) \times L_1(\mu) \longrightarrow L_1(\mu)$ by

$$A(f,g) := \sum_{i=1}^{\infty} \left[\left(\int_{E_i} f d\mu \right) \sum_{\substack{j=1,\\ j\neq i}}^{\infty} \left(\int_{E_j} g d\mu \right) \right] \frac{\chi_{E_1}}{\mu(E_1)}$$

for all $f, g \in L_1(\mu)$. For $f_0^n := \sum_{k=1}^{2n^2} \frac{1}{2n^2} \cdot \frac{\chi_{E_k}}{\mu(E_k)} \in S_{L_1(\mu)}$ and $g_0^n := \sum_{j=1}^{2n^2} \frac{1}{2n^2} \cdot \frac{\chi_{E_j}}{\mu(E_j)} \in S_{L_1(\mu)}$, we see that

$$A(f_0^n, g_0^n) = \sum_{i=1}^{\infty} \left[\left(\sum_{r=1}^{2n^2} \frac{1}{2n^2} \cdot \frac{1}{\mu(E_r)} \int_{E_i} \chi_{E_r} d\mu \right) \sum_{\substack{j=1,\\ j\neq i}}^{\infty} \left(\sum_{s=1}^{2n^2} \frac{1}{2n^2} \cdot \frac{1}{\mu(E_s)} \int_{E_j} \chi_{E_s} d\mu \right) \right] \frac{\chi_{E_1}}{\mu(E_1)}$$
$$= \left(1 - \frac{1}{2n^2}\right) \frac{\chi_{E_1}}{\mu(E_1)}.$$

This shows that $\|A\| = 1$ and so we have $v(A) = \|A\| = 1$ by Theorem 4.1. Choose $n_0 \in \mathbb{N}$ so that $\frac{1}{2n_0^2} < \eta\left(\frac{1}{2}\right)$ and we consider $\chi_\Omega \in S_{L_\infty(\mu)}$. Since $\langle f_0^{n_0}, \chi_\Omega \rangle = \langle g_0^{n_0}, \chi_\Omega \rangle = 1$ and $\langle A(f_0^{n_0}, g_0^{n_0}), \chi_\Omega \rangle = 1 - \frac{1}{2n_0^2}$, there are $B \in \mathcal{L}(^2L_1(\mu); L_1(\mu))$ with $v(B) = 1$, $\overline{f}, \overline{g} \in L_1(\mu)$ and $h \in L_\infty(\mu)$ such that

(a) $|\langle B(\overline{f}, \overline{g}), h \rangle| = 1 = \langle \overline{f}, h \rangle = \langle \overline{g}, h \rangle = \|\overline{f}\|_1 = \|\overline{g}\|_1 = \|h\|_\infty = 1$,
(b) $\|\overline{f} - f_0^{n_0}\|_1 < 1/2$, $\|\overline{g} - g_0^{n_0}\|_1 < 1/2$, $\|h - \chi_\Omega\|_\infty < 1/2$ and $\|B - A\| < \frac{1}{2}$,

Let $a_k := \int_{E_k} |\overline{f}| \chi_{E_k} d\mu$ and $b_k := \int_{F_k} |\overline{g}| \chi_{F_k} d\mu$ for $k \in \mathbb{N} \cup \{0\}$. We have that

$$1 = |\langle B(\overline{f}, \overline{g}), h \rangle| \leq \|h\|_\infty \|B(\overline{f}, \overline{g})\|_1 \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i \cdot b_j = \|\overline{f}\|_1 \|\overline{g}\|_1 = 1.$$

This shows that $\|B(\overline{f} \cdot \chi_{E_i}, \overline{g} \cdot \chi_{E_j})\| = a_i \cdot b_j$ for all $i, j = 0, 1, 2, \ldots$. For $N := \{i \in \mathbb{N} : a_i > 0\}$ and $M := \{j \in \mathbb{N} : b_j > 0\}$, we have that

$$\left\| B\left(\frac{1}{a_i} \overline{f} \cdot \chi_{E_i}, \frac{1}{b_j} \overline{g} \cdot \chi_{E_j} \right) \right\|_1 = 1 \text{ for all } (i,j) \in N \times M. \tag{5.1}$$

We observe that $0 \notin N$. Indeed, if $0 \in N$, then by using (5.1), we have that $\left\| B\left(\frac{1}{a_0} \overline{f} \cdot \chi_{E_0}, \frac{1}{b_j} \overline{g} \cdot \chi_{E_j} \right) \right\|_1 = 1$ for all $j \in M$. On the other hand, $A\left(\frac{1}{a_0} \overline{f} \chi_{E_0}, \frac{1}{b_j} \overline{g} \cdot \chi_{E_j} \right) = 0$. This gives a contradiction because we have $\|A - B\| < \frac{1}{2}$. Similarly we see that $0 \notin M$. Furthermore, we see that $N \cap M = \emptyset$. Indeed, if there exists $i \in N \cap M$, then $A\left(\frac{1}{a_i} \overline{f} \chi_{E_i}, \frac{1}{b_i} \overline{g} \cdot \chi_{E_i} \right) = 0$. Hence, (5.1) provides a contraction since we have $\|A - B\| < \frac{1}{2}$.

Finally, we show that $\|\overline{f} - f_0^{n_0}\|_1 < \frac{1}{2}$ implies $\|\overline{g} - g_0^{n_0}\|_1 \geq \frac{1}{2}$ which is again a contradiction and it finishes our proof. We shall show that the cardinality of $\{1, \ldots, 2n_0^2\} \cap N$ is bigger than n_0^2. Otherwise, there exists a set $S \subset \{j \in \mathbb{N} : 1 \leq j \leq 2n_0^2\}$ with $|S| = n_0^2 + n_1$ for some $n_1 \in \mathbb{N} \cup \{0\}$ such that $a_i = \int_{E_i} |\overline{f}| \chi_{E_i} d\mu = 0$ for all $i \in S$. Then we have that $\|\overline{f} - f_0^{n_0}\|_1 \geq \sum_{j \in S} \int_{E_j} |f_0^{n_0}| \chi_{E_j} d\mu = \frac{1}{2n_0^2} \cdot (n_0^2 + n_1) \geq \frac{1}{2}$. Using the same argument and $N \cap M = \emptyset$, we see that $\|\overline{g} - g_0^{n_0}\|_1 \geq \frac{1}{2}$. ∎

In the next proposition, we deal with an stability result about BPBp-nu for multilinear mappings under direct sums. This is the multilinear version of [27, Lemma 19]. As we did in Lemma **??**, we denote by P_λ the projection from $[\bigoplus_{\lambda\in\Lambda} X_\lambda]$ to X_λ and $\overline{P_\lambda} : X_\lambda \longrightarrow [\bigoplus_{\lambda\in\Lambda} X_\lambda]$ the inclusion. We also use Q_λ and $\overline{Q_\lambda}$ for dual spaces.

Proposition 5.4. *Let $\{X_k : k \in \mathbb{N}\}$ be a family of Banach spaces. Let $X = [\bigoplus_{k=1}^\infty X_k]_{c_0}$ or $X = [\bigoplus_{k=1}^\infty X_k]_{\ell_1}$. If X has the BPBp-nu for multilinear mappings then X_j so does for all $j \in \mathbb{N}$.*

Proof. For $\varepsilon \in (0, 1)$, let $\eta(\varepsilon) > 0$ be the BPBp-nu constant for the space X. Let $A_j \in \mathcal{L}(^N X_j; X_j)$ with $v(A_j) = 1$ and $(x_1^j, \ldots, x_N^j; x_j^*) \in \Pi_N(X_j)$ be such that $|x_j^*(A_j(x_1^j, \ldots, x_N^j))| > 1 - \eta(\varepsilon)$. Define $A \in \mathcal{L}(^N X; X)$ by $A(y_1, \ldots, y_N) := (\overline{P_j} \circ A_j)(P_j(y_1), \ldots, P_j(y_N))$ for all $y_1, \ldots, y_N \in X$. Then $v(A) = v(A_j) = 1$. Consider for each $l = 1, \ldots, N$, the point $x_l := \overline{P_j}(x_l^j) \in S_X$ and $x^* := \overline{Q_j}(x_j^*) \in S_{X^*}$. Then $x^*(x_l) = 1$ for all $l = 1, \ldots, N$ and $|x^*(A(x_1, \ldots, x_N))| = |x_j^*(A_j(x_1^j, \ldots, x_N^j))| > 1 - \eta(\varepsilon)$. Hence, there are $B \in \mathcal{L}(^N X; X)$ with $v(B) = 1$ and $(z_1, \ldots, z_N, z^*) \in \Pi_N(X)$ such that

(a) $|z^*(B(z_1, \ldots, z_N))| = 1$,
(b) $\|z^* - x^*\| < \varepsilon$, $\max_{1\le l\le N} \|z_l - x_l\| < \varepsilon$ and $\|B - A\| < \varepsilon$.

Write $B = (D_1, D_2, \ldots)$ with $D_k \in \mathcal{L}(^N X; X_k)$ for each $k \in \mathbb{N}$. We define the N-linear mapping $B_j \in \mathcal{L}(^N X_j; X_j)$ by $B_j(y_1^j, \ldots, y_N^j) := D_j(\overline{P_j}(y_1^j), \ldots, \overline{P_j}(y_N^j))$ for all $y_1^j, \ldots, y_N^j \in X_j$. Then $v(B_j) \le v(B) = 1$ and $\|B_j - A_j\| \le \|B - A\| < \varepsilon$. Moreover, we have $\|Q_j(z^*) - x_j^*\| \le \|z^* - x^*\| < \varepsilon$ and $\max_{1\le l\le N} \|P_j(z_l) - x_l^j\| \le \max_{1\le l\le N} \|z_l - x_l\| < \varepsilon$. We now need to prove that $(P_j(z_1), \ldots, P_j(z_N), Q_j(z^*)) \in \Pi_N(X_j)$ and $|\langle B_j(P_j(z_1), \ldots, P_j(z_N)), Q_j(z^*)\rangle| = 1$. We consider $X = [\bigoplus_{k=1}^\infty X_k]_{c_0}$ and we omit the proof for the case of ℓ_1-sum since it is similar to this one. Since $(z_1, \ldots, z_N, z^*) \in \Pi_N(X)$, for all $l = 1, \ldots, N$, we have

$$1 = z^*(z_l) = \sum_{n\in\mathbb{N}} Q_n(z^*)P_n(z_l) \le \sum_{n\in\mathbb{N}} \|Q_n(z^*)\|\|P_n(z_l)\|.$$

For $n \ne j$, we have $\|P_n(z_l)\| = \|P_n(z_l) - P_n(x_l)\| \le \|z_l - x_l\|_\infty < \varepsilon$ and so

$$1 \le \sum_{n\in\mathbb{N}} \|Q_n(z^*)\|\|P_n(z_l)\| < \|Q_j(z^*)\|\|P_j(z_l)\| + \varepsilon \sum_{\substack{n\in\mathbb{N},\\ n\ne j}} \|Q_n(z^*)\| \le \|Q_j(z^*)\| + \varepsilon \sum_{\substack{n\in\mathbb{N}\\ n\ne j}} \|Q_n(z^*)\| < \|z^*\|_1 = 1.$$

This implies that $\|Q_j(z^*)\| = 1$ and $Q_n(z^*) = 0$ for all $n \ne j$. Also we see that $Q_j(z^*)(P_j(z_l)) = z^*(z_l) = 1$ for all $l = 1, \ldots, N$. Hence we have $(P_j(z_1), \ldots, P_j(z_N), Q_j(z^*)) \in \Pi_N(X_j)$. Note that we can write

$$z_l = (1 - \varepsilon)\overline{P_j}P_j(z_1) + \varepsilon\left(\overline{P_j}P_j(z_1) + \frac{1}{\varepsilon}\left(z_1 - \overline{P_j}P_j(z_1)\right)\right)$$

for each $l = 1, \ldots, N$. We have that

$$1 = |z^*B(z_1, \ldots, z_N)| \le (1 - \varepsilon)^N |z^*B(\overline{P_j}P_j(z_1), \ldots, \overline{P_j}P_j(z_N))| + \sum_{\substack{\gamma_l\in\{1-\varepsilon,\varepsilon\}\\ l\in\{1,\ldots,N\}\\ \gamma_1\cdots\gamma_N\ne(1-\varepsilon)^N}} \gamma_1\cdots\gamma_N |z^*B(Z_1, \ldots, Z_N)|.$$

where $Z_l = \overline{P_j}P_j(z_l)$ when $\gamma_l = 1 - \varepsilon$ and $Z_l = \overline{P_j}P_j(z_l) + \frac{1}{\varepsilon}\left(z_l - \overline{P_j}P_j(z_l)\right)$ when $\gamma_l = \varepsilon$. Since for every $l = 1, \ldots, N$ we have $Q_j(z^*)(P_j(z_l)) = 1$, $Q_n(z^*) = 0$ and $\|P_n(z_l)\| < \varepsilon$ for $n \ne j$, we deduce $(Z_1, \ldots, Z_N, z^*) \in \Pi_N(X)$ and so $|z^*B(\overline{P_j}P_j(z_1), \ldots, \overline{P_j}P_j(z_N))| \le v(B) = 1$ and $|z^*B(Z_1, \ldots, Z_N)| \le v(B) = 1$. By the equalities

$$(1 - \varepsilon)^N + \sum_{\substack{\gamma_j\in\{1-\varepsilon,\varepsilon\}\\ l\in\{1,\ldots,N\}\\ \gamma_1\cdots\gamma_N\ne(1-\varepsilon)^N}} \gamma_1\cdots\gamma_N = (\varepsilon + (1 - \varepsilon))^N = 1,$$

we have $|z^*B(\overline{P_j}P_j(z_1), \ldots, \overline{P_j}P_j(z_N))| = 1$. Therefore, we see that

$$\begin{aligned} |\langle B_j(P_j(z_1), \ldots, P_j(z_N)), Q_j(z^*)\rangle| &= |\langle D_j(\overline{P_j}P_j(z_1), \ldots, \overline{P_j}P_j(z_N), Q_j(z^*)\rangle| \\ &= |z^*B(\overline{P_j}P_j(z_1), \ldots, \overline{P_j}P_j(z_N))| = 1. \end{aligned}$$

∎

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