A NON-LINEAR BISHOP-PHELPS-BOLLOBÁS TYPE THEOREM

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Abstract. The main aim of this paper is to prove a Bishop-Phelps-Bollobás type theorem on the unital uniform algebra $\mathcal{A}_{w^*u}(B_{X^*})$ consisting of all w^*-uniformly continuous functions on the closed unit ball B_{X^*} which are holomorphic on the interior of B_{X^*}. We show that this result holds for $\mathcal{A}_{w^*u}(B_{X^*})$ if X^* is uniformly convex or X^* is the uniformly complex convex dual space of an order continuous absolute normed space. The vector-valued case is also studied. Throughout the paper we consider complex Banach spaces.

In 1961, Bishop and Phelps proved that the set of norm attaining functionals is dense in the dual space [5]. They questioned whether the same result holds for bounded linear operators and the answer was given two years later by Lindenstrauss [17] who gave a counterexample proving that in general the answer is false. However, he also presented conditions on a Banach space to get positive results. On the other hand, Bollobás proved a stronger version for linear functionals which is nowdays known as the Bishop-Phelps-Bollobás theorem [6]. This says that functionals and points which they almost attain their norms can be simultaneously approximated by norm attaining functionals and points which they attain their norms. We highlight this theorem because it is the main motivation for the present work. If X is a Banach space, $\varepsilon \in (0,2)$ and $(x,x^*) \in B_X \times B_{X^*}$ satisfy the following inequality

$$\operatorname{Re} x^*(x) > 1 - \frac{\varepsilon^2}{2},$$

then there is $(y,y^*) \in S_X \times S_{X^*}$ such that

$$|y^*(y)| = 1, \quad \|y - x\| < \varepsilon \text{ and } \|y^* - x^*\| < \varepsilon$$

(this version can be found in [10, Corollary 2.4]).

Since 2008, with the seminal paper by Acosta, Aron, García and Maestre [2], a lot of attention had been paid in the attempt to get Bishop-Phelps-Bollobás type theorems for bounded linear operators, homogeneous polynomials and multilinear mappings by putting conditions on the Banach spaces as Lindenstrauss did. Our aim here is to get a Bishop-Phelps-Bollobás type theorem for holomorphic functions. In [1] the authors showed that if X is a complex Banach space with Radon-Nikodým property and if $\mathcal{A}_u(B_X)$ stands for the space of all uniformly continuous functions on the closed unit ball B_X which are holomorphic on the interior endowed with the supremum norm, then the set of all norm attaining elements is dense in $\mathcal{A}_u(B_X)$. This result was sharpened to the denseness of the set of all strong peak functions on $\mathcal{A}_u(B_X)$ [13, Theorem 4.4].

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Before we give our results, we present the necessary background. Throughout the paper we consider Banach spaces over the complex field. We will use the notation X^*, S_X and B_X for the dual space, the unit sphere and the closed unit ball of X, respectively. We denote by $\mathcal{A}_{w^*u}(B_{X^*})$ the unital uniform algebra of all w^*-uniformly continuous functions from B_{X^*} into $\mathbb{C}$ which are holomorphic on the interior of B_{X^*} endowed with the supremum norm

$$\|f\|_\infty = \sup\{|f(x^*)| : x^* \in B_{X^*}\}.$$

It is known that $\mathcal{A}_{w*}(B_{X^*})$ coincides with $\mathcal{A}_{w*u}(B_{X^*})$ [4]. An element $x_0^* \in S_{X^*}$ is said to be a *strong peak point* for $\mathcal{A}_{w^*u}(B_{X^*})$ if there is $f \in \mathcal{A}_{w^*u}(B_{X^*})$ such that $\|f\|_\infty = |f(x_0^*)| = 1$ and for every w^*-neighborhood W of x_0^*, we have

$$\sup_{x^* \in B_{X^*} \setminus W} |f(x^*)| < 1.$$

In this case, f is said to be a *strong peak function at* x_0^*. We denote by Γ the set of all strong peak points for $\mathcal{A}_{w^*u}(B_{X^*})$.

Let K be a nonempty compact Hausdorff space and let A be a uniform algebra. Given $t \in K$, the function $\delta_t : A \longrightarrow \mathbb{C}$ defined by $\delta_t(f) = f(t)$ is called the *evaluation functional* at t. A set $S \subset K$ is said to be a boundary for the uniform algebra A if for every $f \in A$, there is $x \in S$ such that $|f(x)| = \|f\|_\infty$. If $S = \{x^* \in A^* : \ \|x^*\| = x^*(\mathbf{1}) = 1\}$ and $\mathrm{Ext}_{\mathbb{R}}(S)$ stands for the set of all real extreme points of S, then $\Gamma_0(A) = \{t \in K : \delta_t \in \mathrm{Ext}_{\mathbb{R}}(S)\}$ is a boundary for A which is called the *Choquet boundary* of A.

Let A be a unital uniform algebra on K. It is well-known that the Choquet boundary for A consists exactly the set of all strong peak points for A if K is metrizable [14, Theorem 4.3.5, Corollary 4.3.7]. Recall that an element x of the unit ball of a Banach space X is said to be a *complex extreme point* if $\sup_{0\leqslant\theta\leqslant 2\pi} \|x + e^{i\theta}y\| > 1$ for all nonzero $y \in X$. We denote by $\mathrm{Ext}_{\mathbb{C}}(B_X)$ the set of all complex extreme points of B_X. Note that Γ is the Choquet boundary for $\mathcal{A}_{w^*u}(B_{X^*})$ and Γ is contained in $\mathrm{Ext}_{\mathbb{C}}(B_{X^*})$. In particular, it is observed that $\Gamma = \mathrm{Ext}_{\mathbb{C}}(B_{X^*})$ for finite dimensional Banach spaces X [11, Proposition 1.1].

To get a version of the Bishop-Phelps-Bollobás theorem on the space $\mathcal{A}_{w^*u}(B_{X^*})$, we need to consider stronger peak functions. A point $x_0^* \in S_{X^*}$ is said to be a *strong peak point for* $\mathcal{A}_{w^*u}(B_{X^*})$ *with respect to the norm* if there is $f \in \mathcal{A}_{w^*u}(B_{X^*})$ such that $\|f\|_\infty = |f(x_0^*)| = 1$ and for all $\delta > 0$,

$$\sup_{y^* \in B_{X^*} \setminus B(x_0^*, \delta)} |f(y^*)| < 1.$$

Equivalently, it happens when $\|f\|_\infty = |f(x_0^*)| = 1$ and $\{x_n^*\} \xrightarrow{\|\cdot\|} x_0^*$ whenever $\{x_n^*\} \subset B_{X^*}$ satisfies $\lim_{n\to\infty} |f(x_n^*)| = 1$. The function f is called a *strong peak function at* x_0^* *with respect to the norm*. We denote by Γ_s the set of all strong peak points for $\mathcal{A}_{w^*u}(B_{X^*})$ with respect to the norm. We note that for every w^*-neighborhood W of x_0^*, there exists $\delta > 0$ such that $B(x_0^*, \delta) \subset W$. This implies that $\Gamma_s \subset \Gamma$.

In what follows, we present a non-linear version of the Bishop-Phelps-Bollobás theorem for the unital uniform algebra $\mathcal{A}_{w^*u}(B_{X^*})$ by assuming that the set of all strong peak points with respect to the norm is norm dense in the unit sphere of the dual space. After that, we give some consequences like a Lindenstrauss-Bollobás type theorem as well as a vector-valued extension of the main result. Finally, we present some examples when Γ_s is norm dense in S_{X^*}.

We start with the Bishop-Phelps-Bollobás theorem for $\mathcal{A}_{w^*u}(B_{X^*})$.

Theorem 1. Let X be a complex Banach space and suppose that Γ_s is norm dense in S_{X^*}. Then, given $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that whenever $f \in \mathcal{A}_{w^*u}(B_{X^*})$ with $\|f\|_\infty = 1$ and $x_0^* \in S_{X^*}$ satisfy

$$|f(x_0^*)| > 1 - \eta(\varepsilon),$$

there are $g \in \mathcal{A}_{w^*u}(B_{X^*})$ with $\|g\|_\infty = 1$ and $x_1^* \in S_{X^*}$ such that

$$|g(x_1^*)| = 1, \quad \|g - f\|_\infty < \varepsilon \quad \text{and} \quad \|x_1^* - x_0^*\| < \varepsilon.$$

The following Urysohn type lemma plays an important role in the proof of the theorem.

Lemma 2. [9, Lemma 2.7] Let $A \subset C(K)$ be a unital uniform algebra and Γ_0 its Choquet boundary. Then, for any open subset U of K with $U \cap \Gamma_0 \neq \emptyset$ and for $0 < \varepsilon < 1$, there exist $f \in A$ and $t_0 \in U \cap \Gamma_0$ satisfying

(i) $f(t_0) = \|f\|_\infty = 1$,
(ii) $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and
(iii) $|f(t)| + (1-\varepsilon)|1 - f(t)| \leqslant 1$ for every $t \in K$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let $\varepsilon \in (0,1)$ be given and define $\eta(\varepsilon) := \frac{\varepsilon}{4} > 0$. Let $f \in \mathcal{A}_{w^*u}(B_{X^*})$ with $\|f\|_\infty = 1$ and $x_0^* \in S_{X^*}$ be such that

$$|f(x_0^*)| > 1 - \eta(\varepsilon).$$

Note that

$$U_1 = \left\{ x^* \in B_{X^*} : \left| \frac{f(x_0^*)}{|f(x_0^*)|} - f(x^*) \right| < \eta(\varepsilon) \right\}$$

is a nonempty w^*-open set on B_{X^*}. Since Γ_s is norm dense in S_{X^*}, there is $z_0^* \in \Gamma_s$ such that

$$z_0^* \in U_1 \quad \text{and} \quad \|z_0^* - x_0^*\| < \frac{\varepsilon}{2}.$$

Let $h \in \mathcal{A}_{w^*u}(B_{X^*})$ be a strong peak function at z_0^* with respect to the norm. Then $\|h\|_\infty = |h(z_0^*)| = 1$ and

$$r := \sup_{y^* \in B_{X^*} \setminus B\left(z_0^*, \frac{\varepsilon}{2}\right)} |h(y^*)| < 1. \tag{1}$$

Now define

$$U_2 := U_1 \cap \{x^* \in B_{X^*} : |h(x^*)| > r\}$$

which is a w^*-open set on B_{X^*}. Note that Γ_s is contained in the Choquet boundary of $\mathcal{A}_{w^*u}(B_{X^*})$. Since $z_0^* \in U_2 \cap \Gamma_s$, we can apply Lemma 2 to get $\phi \in \mathcal{A}_{w^*u}(B_{X^*})$ and $x_1^* \in U_2$ satisfying the following conditions:

(i) $\phi(x_1^*) = \|\phi\|_\infty = 1$,
(ii) $|\phi(x^*)| < \eta(\varepsilon)$ for all $x^* \in B_{X^*} \setminus U_2$ and
(iii) $|\phi(x^*)| + (1 - \eta(\varepsilon))|1 - \phi(x^*)| \leqslant 1$ for all $x^* \in B_{X^*}$.

Now let, for $x^* \in B_{X^*}$,

$$g(x^*) := \frac{f(x_0^*)}{|f(x_0^*)|}\phi(x^*) + (1-\eta(\varepsilon))(1-\phi(x^*))f(x^*).$$

Then $g \in \mathcal{A}_{w^*u}(B_{X^*})$ and, by (i), we have $|g(x_1^*)| = \phi(x_1^*) = 1$. By using (iii), for all $x^* \in B_{X^*}$, we get that

$$|g(x^*)| \leqslant |\phi(x^*)| + (1-\eta(\varepsilon))|1-\phi(x^*)| \leqslant 1.$$

This shows that $\|g\|_\infty = |g(x_1^*)| = 1$. Since $x_1^* \in U_2$, we have that $|h(x_1^*)| > r$ and using (1), $\|x_1^* - z_0^*\| < \frac{\varepsilon}{2}$. This implies that

$$\|x_1^* - x_0^*\| \leqslant \|x_1^* - z_0^*\| + \|z_0^* - x_0^*\| < \varepsilon.$$

Finally we show that $\|g - f\|_\infty < \varepsilon$. Indeed, we write

$$g - f = \left(\frac{f(x_0^*)}{|f(x_0^*)|} - f\right)\phi - \eta(\varepsilon)(1-\phi)f.$$

If $x^* \in U_2$, then

$$\|g(x^*) - f(x^*)\| < \|\phi\|_\infty\eta(\varepsilon) + \eta(\varepsilon)(1 + \|\phi\|_\infty)\|f\|_\infty = 3\eta(\varepsilon) < \varepsilon.$$

If $x^* \in B_{X^*} \setminus U_2$, then we use (ii) to get

$$\|g(x^*) - f(x^*)\| \leqslant 2|\phi(x^*)| + \eta(\varepsilon)(1 + |\phi(x^*)|)\|f\|_\infty < 4\eta(\varepsilon) = \varepsilon.$$

Therefore, we have $\|g - f\|_\infty < \varepsilon$ and this finishes the proof. □

In [8, Theorem B], the authors proved that if X is a Banach space whose dual is separable and has the approximation property, then the set of analytic functions whose Aron-Berner extensions attain their norm is dense in $\mathcal{A}_u(B_X)$. On the other hand, [7, Proposition 4.4] gives us a counterexample of a space which does not satisfy a Lindenstrauss-Bollobás type theorem for multilinear forms or multilinear mappings. Here, since $\mathcal{A}_{wu}(B_X)$ is isometrically isomorphic to $\mathcal{A}_{w^*u}(B_{X^{**}})$ (see [4, Theorem 6.3]), we have the following consequence of Theorem 1.

Proposition 3. Let X be a complex Banach space and let $\Gamma_s(B_{X^{**}})$ be the set of all strong peak points for $A_{w^*u}(B_{X^{**}})$ with respect to the norm. Suppose that $\Gamma_s(B_{X^{**}})$ is dense in $S_{X^{**}}$. Then, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $f \in \mathcal{A}_{wu}(B_X)$ with $\|f\|_\infty = 1$ and $x_0 \in S_X$ satisfy

$$|f(x_0)| > 1 - \eta(\varepsilon),$$

there are $g \in S_{\mathcal{A}_{wu}(B_X)}$ and $z_0^{**} \in S_{X^{**}}$ such that

$$|ABg(z_0^{**})| = 1, \quad \|g - f\|_\infty < \varepsilon \quad \text{and} \quad \|z_0^{**} - x_0\| < \varepsilon,$$

where $ABg \in \mathcal{A}_{w^*u}(B_{X^{**}})$ stands for the Aron-Berner extension of g to $B_{X^{**}}$.

Let Y be a complex Banach space. We denote by $\mathcal{A}_{w^*u}(B_{X^*}, Y)$ the space of all Y-valued holomorphic functions on the interior of B_{X^*} which are w^*-to-norm uniformly continuous on B_{X^*} equipped with the supremum norm. Theorem 1 can be extended to the vector-valued case as follows by using the same proof.

Theorem 4. Let X and Y be complex Banach spaces and suppose that Γ_s is norm dense in S_{X^*}. Then, given $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that whenever $f \in \mathcal{A}_{w^*u}(B_{X^*}, Y)$ with $\|f\|_\infty = 1$ and $x_0^* \in S_{X^*}$ satisfy

$$\|f(x_0^*)\|_Y > 1 - \eta(\varepsilon),$$

there are $g \in \mathcal{A}_{w^*u}(B_{X^*}, Y)$ with $\|g\|_\infty = 1$ and $x_1^* \in S_{X^*}$ such that

$$\|g(x_1^*)\|_Y = 1, \quad \|g - f\|_\infty < \varepsilon \quad \text{and} \quad \|x_1^* - x_0^*\| < \varepsilon.$$

We observe that there are Banach spaces Y such that the Bishop-Phelps-Bollobás theorem does not hold for bounded linear operators from ℓ_1 into Y (see [2, Remark 2.4]). Moreover, in [3, Corollary 3.3], it was shown that such theorem for operators from ℓ_1^2 into some Banach space Y is also not true where ℓ_p^n is $\mathbb{R}^n$ with ℓ_p norm for $n \in \mathbb{N}$. Nevertheless, we will prove that if $X^* = \ell_1^n$ or ℓ_1, then Theorem 4 holds for all complex Banach spaces Y (see Proposition 5 below). It is worth mentioning that in Bishop-Phelps-Bollobás type theorems we consider norm-to-norm continuous operators and here we are considering w^*-to-norm continuous functions. These two continuities coincide when the domain is finite dimensional.

Next we present some spaces satisfying the condition that Γ_s is norm dense in S_{X^*}. Let X be a subspace of $\mathbb{C}^J$ for a set J. If $x = (x(j))_{j\in J}$ is an element of $\mathbb{C}^J$, the absolute value $|x|$ is defined to be $|x| = (|x(j)|)_{j\in J}$. Let $x = (x(j))_{j\in J}$ and $y = (y(j))_{j\in J}$. We say that $x \leqslant y$ whenever $x, y \in \mathbb{R}^J$ and $x(j) \leqslant y(j)$ for all j. A norm $\|\cdot\|$ on X is said to be an absolute norm if $(X, \|\cdot\|)$ is a Banach space and, if $x \in X$ and $|y| \leqslant |x|$ for some $y \in \mathbb{C}^J$ we have that $y \in X$ and $\|y\| \leqslant \|x\|$. In this case, we call X as an absolute normed space. We denote by e_j the j-th standard unit vector defined by $e_j(i) = 0$ if $i \neq j$ and $e_j(j) = 1$ for all j. We also assume that absolute normed spaces contain each e_j and $\|e_j\| = 1$. Notice that absolute normed spaces are complex Banach lattices they can be viewed as Köthe spaces on the measure space $(J, 2^J, \nu)$ where ν is the counting measure on a set J. We say that a Banach lattice X is order continuous if every downward directed set $\{x_\alpha\}$ in X with $\bigwedge_\alpha x_\alpha = 0$, $\lim_\alpha \|x_\alpha\| = 0$. For the reference about the order continuity and Köthe spaces, see [18].

It is well-known that an absolute normed space X is order continuous if and only if X is the closed linear span of the set $\{e_j : j \in J\}$ of all standard unit vector basis (for the reference of this fact, see the proof of Proposition 7.1 in [15].) If X is order continuous, then X^* is also an absolute normed space [18, P. 29]. In fact, the dual space X^* can be identified with the Köthe dual X' consisting of all $x^* = (x^*(j))_{j\in J}$ in $\mathbb{C}^J$ such that

$$\|x^*\| = \sup \left\{ \left| \sum_{j\in J} x^*(j)x(j) \right| : x = (x(j))_{j\in J} \in S_X \right\} < \infty.$$

The standard unit vector in X^* will be denoted by e_j^*.

Given a complex Banach space X, the *modulus of complex convexity* is defined by

$$H(\varepsilon) = \inf \left\{ \sup_{0\leqslant\theta\leqslant 2\pi} \|x + e^{i\theta}y\| - 1 : x \in S_X, \|y\| \geq \varepsilon \right\}.$$

A Banach space is said to be *uniformly complex convex* whenever $H(\varepsilon) > 0$ for all $\varepsilon > 0$. The complex convexity and monotonicity is closely related in Banach lattices. Recall that a Banach space is said to be *uniformly monotone* if, for all $\varepsilon > 0$, we have

$$M(\varepsilon) = \inf \{ \||x| + |y|\| - 1 : \ x \in S_X, \ \|y\| \geqslant \varepsilon \} > 0.$$

It is shown in [16, Theorem 3.5] that a complex Banach lattice (an absolute normed space) is uniformly complex convex if and only if it is uniformly monotone. It is also shown [16, Proposition 2.2] that if a complex Banach lattice is uniformly monotone, then it is order continuous. We will use these facts in the next result. It is known that ℓ_p and ℓ_p^n are uniformly monotone (and so is uniformly complex convex) for $1 \leqslant p < \infty$. The uniform complex convexity of Orlicz-Lorentz spaces is characterized in [12].

Proposition 5. Suppose that X is an order continuous subspace of $\mathbb{C}^J$ with an absolute norm and suppose that X^* is uniformly complex convex. Let F be a finite dimensional subspace spanned by a finite number of standard unit vectors of X^*. Then S_F is contained in Γ_s and Γ_s is norm dense in S_{X^*}.

Proof. Suppose that X^* is uniformly complex convex. Then it is uniformly monotone. By the above discussion, X^* is order continuous. So, X^* is the norm closure of the span of all standard unit vectors e_j^* in X^*. Thus, it is enough to show that S_F is contained in Γ_s.

Suppose that F is the linear span of $\{e_j^* : j \in J_0\}$, where J_0 is a finite subset of J. Let $P : X^* \longrightarrow F$ be the natural projection defined by $P(x^*) = \sum_{j \in J_0} x^*(e_j)e_j^*$. Note that P is w^*-to-norm continuous and $\|x^*\| = \||Px^*| + |(I-P)x^*|\|$ for all $x^* \in X^*$. Let $x_0^* \in S_F$. Since X^* is uniformly complex convex, $S_F = \operatorname{Ext}_{\mathbb{C}}(B_F)$ and this is exactly the set of all strong peak points for $A_u(B_F)$ because F is finite dimensional [11, Proposition 1.1]. So, there is a strong peak function g at x_0^* in $\mathcal{A}_u(B_F)$.

We will show that the function $f : B_{X^*} \longrightarrow \mathbb{K}$ defined by

$$f(x^*) := (g \circ P)(x^*) \quad (x^* \in B_{X^*})$$

is a strong peak function for $\mathcal{A}_{w^*u}(B_{X^*})$ at x_0^* with respect to the norm and the proof will be done. Note first that $f \in \mathcal{A}_{w^*u}(B_{X^*})$. To show that it is a strong peak function at x_0^* with respect to the norm, assume that there are $\delta_0 > 0$ and a sequence $\{x_k^*\} \subset B_{X^*}$ such that $\lim_{k\to\infty} |f(x_k^*)| = 1$ but $\|x_k^* - x_0^*\| \geqslant \delta_0$ for every $k \in \mathbb{N}$. Since $\lim_{k\to\infty} |g(P(x_k^*))| = 1$ and g is a strong peak function at x_0^* in $A_u(B_F)$, we get that $\lim_{k\to\infty} \|Px_k^* - x_0^*\| = 0$. Since $\|(Px_k^* - x_0^*) + (I-P)(x_k^*)\| = \|x_k^* - x_0^*\| \geqslant \delta_0$, we may assume that $\|(I-P)(x_k^*)\| \geq \delta_0/2$ for all k. Nevertheless, for all $k \in \mathbb{N}$, we get that

$$\begin{aligned}
1 \geq \|x_k^*\| &= \|P(x_k^*) + (I-P)(x_k^*)\| \\
&\geqslant \|P(x_0^*) + (I-P)(x_k^*)\| - \|x_0^* - Px_k^*\| \\
&= \||P(x_0^*)| + |(I-P)(x_k^*)|\| - \|x_0^* - Px_k^*\| \\
&\geqslant 1 + M(\delta_0/2) - \|x_0^* - Px_k^*\|.
\end{aligned}$$

This shows that $M(\delta_0/2) \leqslant 0$ which is a contradiction since X^* is uniformly monotone. □

Recall that $x_0^* \in B_{X^*}$ is said to be a *w^*-strongly exposed point* if there exists $x_0 \in S_X$ such that $x_0^*(x_0) = 1$ and $\lim_{\delta\to 0+} \operatorname{diam}(S(x_0,\delta)) = 0$, where $S(x_0,\delta) = \{x^* \in B_{X^*} : \operatorname{Re} x^*(x_0) > 1-\delta\}$. We have the following proposition.

Proposition 6. Let X be a complex Banach space. If $x_0^* \in B_{X^*}$ is a w^*-strongly exposed point, then x_0^* is a strong peak point for $\mathcal{A}_{w^*u}(B_{X^*})$ with respect to the norm. In particular, if X is reflexive and X^* is locally uniformly convex, then $\Gamma_s = S_{X^*}$.

Proof. Suppose that $x_0^* \in S_{X^*}$ is a w^*-strongly exposed point. Then there exists $x_0 \in S_X$ such that $\operatorname{Re} x_0^*(x_0) = 1$ and for every $\{x_n^*\} \subset B_{X^*}$ with $\operatorname{Re} x_n^*(x_0) \longrightarrow 1$, we have that $\|x_n^* - x_0^*\| \longrightarrow 0$ when $n \to \infty$.

Define $f : B_{X^*} \longrightarrow \mathbb{C}$ by

$$f(x^*) := \frac{1 + x^*(x_0)}{2} \quad (x^* \in B_{X^*}). \tag{2}$$

Then $f \in \mathcal{A}_{w^*u}(B_{X^*})$, $\|f\|_\infty \leqslant 1$ and

$$\|f\|_\infty \geqslant \frac{1 + \operatorname{Re} x_0^*(x_0)}{2} = 1.$$

Hence, $\|f\|_\infty = f(x_0^*) = 1$. Let $\{x_n^*\} \subset B_{X^*}$ satisfy that $|f(x_n^*)| \longrightarrow 1$ as $n \to \infty$ which means $\left|\frac{1+x_n^*(x_0)}{2}\right| \longrightarrow 1$. Since

$$|1 + x_n^*(x_0)|^2 + |1 - x_n^*(x_0)|^2 = 2(1 + |x_n^*(x_0)|^2),$$

we have that

$$|1 - x_n^*(x_0)|^2 \leqslant 4 - |1 + x_n^*(x_0)|^2 \longrightarrow 0 \quad (n \to \infty).$$

This implies that $\operatorname{Re} x_n^*(x_0) \longrightarrow 1$. Hence $\|x_n^* - x_0^*\| \longrightarrow 0$ when $n \to \infty$. This shows that x_0^* is a strong peak point for $\mathcal{A}_{w^*u}(B_{X^*})$ with respect to the norm.

Now suppose that X is reflexive and X^* is locally uniformly convex. We will prove that every point $x_0^* \in S_{X^*}$ is a w^*-strongly exposed point. Indeed, since X is reflexive, there exists $x_0 \in S_X$ such that $\operatorname{Re} x_0^*(x_0) = 1$. Let $\{x_n^*\} \subset B_{X^*}$ be such that $\operatorname{Re} x_n^*(x_0) \longrightarrow 1$ when $n \to \infty$. Then,

$$1 \geqslant \left\|\frac{x_n^* + x_0^*}{2}\right\| \geqslant \operatorname{Re}\left(\frac{x_n^*(x_0) + x_0^*(x_0)}{2}\right) \overset{n\to\infty}{\longrightarrow} 1.$$

Since X^* locally uniformly convex, we get that $\|x_n^* - x_0^*\| \longrightarrow 0$ when $n \to \infty$. By the previous paragraph, $\Gamma_s = S_{X^*}$ as desired. ☐

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