CALCULUS 1 (BE5B01MA1)
FINAL EXAM 4 (2020, FEB, 13TH)

⋆ *This exam has* **6 problems** *with a total of* **100 points**.
⋆ *Grades classification*: **F** ($\leqslant$49pts), **E** (50-59), **D** (60-69), **C** (70-79), **B** (80-89), **A** (90-100).

NAME: ____

Exercise 1. [20pts] Consider the function $f(x) = \dfrac{x}{x-1}$.

(a) [1pt] Determine the domain of f and evaluate $f(0)$.

(b) Compute

(i) [1pt] $\lim_{x\to 1^+} f(x)$.

(ii) [1pt] $\lim_{x\to +\infty} f(x)$.

(iii) [1pt] $\lim_{x\to 1^-} f(x)$.

(iv) [1pt] $\lim_{x\to -\infty} f(x)$.

(c) [5pts] Find $f'(x)$ and the intervals where f is increasing and decreasing.

(d) [5pts] Find $f''(x)$ and the intervals where f is concave upward and downward.

(e) [5pts] Use all the previous information to sketch the graph of f.

Exercise 2. [15pts]

(a) [5pts] Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

(b) [5pts] Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(c) [5pts] Find now the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n(n+1)} - \frac{1}{2^n}\right)$.

Exercise 3. [15pts] Evaluate the following integrals.

(a) [5pts] $\displaystyle\int e^x \sin x dx$.

(b) [5pts] $\displaystyle\int_1^2 \frac{1}{(3-5x)^2} dx$.

(c) [5pts] $\displaystyle\int \tan^3(x) dx$.

Exercise 4. [20pts]

(a) [5pts] Find the Maclaurin series for $f(x) = \cos(x)$.

(b) [5pts] Find its radius of convergence.

(c) [5pts] Use (a) to find the Maclaurin series for the function $g(x) = x\cos(x)$.

(d) [5pts] Use (c) to show that $\lim_{x\to 0} \frac{x\cos(x) - x + \frac{x^3}{2}}{x^5} = \frac{1}{24}$.

Exercise 5. [20pts] Decide if the following statements are **true** or **false**.

- If it is **true**, explain why.
- If it is **false**, give an example that disproves the statement or explain why it is not true.

(a) [4pts] If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) [4pts] The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ is $R = 1/3$.

(c) [4pts] The Ratio Test can be used to determine whether $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

(d) [4pts] If $\int_0^1 f(x)dx = 0$, then $f(x) = 0$ for $0 \leqslant x \leqslant 1$.

(e) [4pts] If $f'(c) = 0$, then f has a local maximum or minimum at c.

Exercise 6. [10pts] Give the definition of

(a) [2pts] a *continuous function* at a point a.

(b) [2pts] a *differentiable function* at a point a.

(c) [2pts] the *Maclaurin series* of a function f at a point a.

(d) [2pts] a *bounded sequence* $(a_n)_{n=1}^{\infty}$.

(e) [2pts] a *power series*.

Exercise 1:

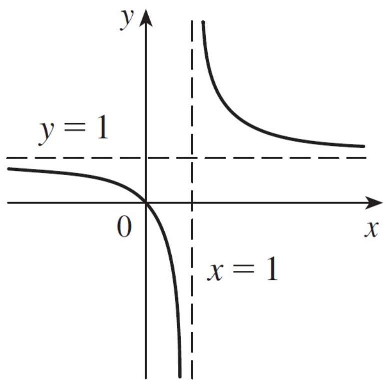
(a) $D(f) = \{x \in \mathbb{R} : x \neq 1\}$ and $f(0) = 0$.

(b) (i) $+\infty$; (ii) 1; (iii) $-\infty$; (iv) 1.

(c) $f'(x) = -\dfrac{1}{(x-1)^2}$. Since 1 is not in the domain of f, f is decreasing on $(-\infty, 1) \cup (1, \infty)$.

(d) $f''(x) = \dfrac{2}{(x-1)^3}$. Since $f''(x) > 0$ when $x > 1$ and $x \neq 1$ and $f''(x) < 0$ when $x < 1$ and $x \neq 1$, we have that f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$.

(e)



Exercise 2:

(a) Notice that for every $n \in \mathbb{N}$, we have

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

So,

$$s_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1}.$$

This shows that $\lim_{n\to\infty} s_n = 1$. So, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

(b) It is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$. So,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

(c) Since both $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ are convergent, we have that

$$\sum_{n=1}^{\infty} \left(\frac{2}{n(n+1)} - \frac{1}{2^n} \right) = 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \sum_{n=1}^{\infty} \frac{1}{2^n} = 2 \cdot 1 - 1 = 1.$$

Exercise 3:

(a) By parts twice (first $u = e^x$ and $dv = \sin(x)dx$ and then $u = e^x$ and $dv = \cos(x)dx$). The final answer is

$$\int e^x \sin(x)dx = \frac{1}{2}e^x(\sin(x) - \cos(x)) + C.$$

(b) By substitution (use $u = 3 - 5x$) we get that

$$\int \frac{1}{(3-5x)^2}dx = \frac{1}{15-25x}dx,$$

which implies that

$$\int_1^2 \frac{1}{(3-5x)^2}dx = \frac{1}{14}.$$

(c) We have that $\tan^3(x) = \tan(x)\tan^2(x) = \tan(x)(\sec^2(x) - 1) = \tan(x)\sec^2(x) - \tan(x)$. So,

$$\int \tan^3(x)dx = \int \tan(x)\sec^2(x)dx - \int \tan(x)dx.$$

In the first integral, we use $u = \tan(x)$. Then, we get the final result:

$$\int \tan^3(x)dx = \frac{\sec^2(x)}{2} + \ln|\cos(x)| + C.$$

Exercise 4:

(a) Let $f(x) = \cos(x)$. Then, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and so on. Then, we have that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$, $f^{(4)}(0) = 1$, and so on. Then, the Maclaurin series of $\cos(x)$ is

$$1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.$$

(b) If $a_n = (-1)^n\frac{x^{2n}}{(2n)!}$, then $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow 0$ as $n \to \infty$. So, $R = \infty$.

(c) We have that

$$x\cos(x) = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!} = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \ldots.$$

(d) By (c), we have that

$$x\cos(x) - x + \frac{x^3}{2} = \frac{x^5}{4!} - \frac{x^7}{6!} + \ldots.$$

So,

$$\lim_{x\to 0}\frac{x\cos(x) - x + \frac{x^3}{2}}{x^5} = \frac{1}{4!} = \frac{1}{24}.$$

Exercise 5:

(a) (F) If $a_n = \frac{1}{n}$, then $\lim_{n\to\infty} a_n = 0$ but $\sum a_n$ diverges.

(b) (V) If $a_n = (-3)^n x^n/\sqrt{n+1}$, then $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow 3|x|$ as $n \to \infty$. By the Ratio Test, the series converges if $3|x| < 1$ and diverges $3|x| > 1$. This means that $R = 1/3$.

(c) (F) The Ratio Test is inconclusive since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

(d) (F) For instance, let $f(x) = x - \frac{1}{2}$. Then,

$$\int_0^1 f(x)dx = \left[\frac{x^2}{2} - \frac{1}{2}x\right]_0^1 = 0$$

and $f(x) \neq 0$ on $(0,1)$.

(e) (F) Consider, for instance, $f(x) = x^3$. We have that $f'(0) = 0$ but f does not have neither a local maximum nor a local minimum at 0.