Calculus 1 (BE5B01MA1)
Final Exam 5 (2020, Feb, 19th)

⋆ *This exam has* **6 problems** *with a total of* **100 points**.
⋆ *Grades classification:* **F** (⩽49pts), **E** (50-59), **D** (60-69), **C** (70-79), **B** (80-89), **A** (90-100).

Name: ____

Exercise 1. [20pts] Consider the function $f(x) = x\sqrt{6-x}$.

(a) [1pt] Determine the domain of f and evaluate $f(0)$.

(b) [8pts] Find $f'(x)$ and the intervals where f is increasing and decreasing.

(c) [8pts] Find $f''(x)$ and the intervals where f is concave upward and downward.

(d) [3pts] Use the previous information to sketch the graph of f.

Exercise 2. [15pts] Show that $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n^2+3n-2} = \frac{1}{2}$.

Exercise 3. [15pts] Calculate.

(a) [5pts] $\displaystyle\int \ln(2x+1)dx$.

(b) [5pts] $\displaystyle\lim_{t\to-\infty} te^t$.

(c) [5pts] $\displaystyle\int_{-\infty}^{0} xe^x dx$.

Exercise 4. [20pts]

(a) [5pts] Find the Maclaurin series for $\sin(x)$.

(b) [5pts] Find its radius of convergence.

(c) [5pts] Evaluate $\displaystyle\lim_{x\to0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$ using the L'Hospital's Rule.

(d) [5pts] Evaluate $\displaystyle\lim_{x\to0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$ using the Maclaurin series for $\sin(x)$.

Exercise 5. [20pts] Decide if the following statements are **true** or **false**.

- If it is **true**, explain why.
- If it is **false**, give an example that disproves the statement or explain why it is not true.

(a) [4pts] If f, g are continuous on $[a, b]$, then $\int_a^b [f(x)g(x)]dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$.

(b) [4pts] $\int xf(x)dx = x\int f(x)dx$.

(c) [4pts] $\int_{-1}^{1}(ax^2 + bx + c)dx = 2\int_0^1 (ax^2 + c)dx$.

(d) [4pts] If $\int_0^1 f(x)dx = 0$, then $f(x) = 0$ for $0 \leqslant x \leqslant 1$.

(e) [4pts] If $f'(c) = 0$, then f has a local maximum or minimum at c.

Exercise 6. [10pts] Give the definition of

(a) [2pts] a *continuous function* at a point a.

(b) [2pts] a *differentiable function* at a point a.

(c) [2pts] the *Maclaurin series* of a function f at a point a.

(d) [2pts] a *bounded sequence* $(a_n)_{n=1}^{\infty}$.

(e) [2pts] a *power series*.

Solutions

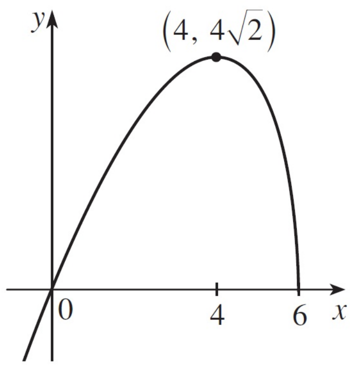
Exercise 1:

(a) $D(f) = \{x \in \mathbb{R} : x \leqslant 6\}$ and $f(0) = 0$.

(b) $f'(x) = \dfrac{3(4-x)}{2\sqrt{6-x}}$. f is increasing on $(-\infty, 4)$ and decreasing on $(4, 6)$

(c) $f''(x) = \dfrac{3(x-8)}{4(6-x)^{3/2}}$. f is concave downward on $(-\infty, 6)$.
(Remember that $x = 8$ is **not** in the domain of f)

(d)



Exercise 2: By using Partial Fractions, we have that

$$\frac{1}{9n^2+3n-2} = \frac{1}{3n-1} - \frac{1}{3n+2}.$$

Now, we are dealing with a telescopic series.

Exercise 3:

(a) Put $t = 2x+1$. Then, $dt = 2dx$ and

$$\int \ln(2x+1)dx = \frac{1}{2}\int \ln(t)dt.$$

Now, we solve the last integral by parts with $u = \ln(t)$ and $dv = dt$. The final result is:

$$\int \ln(2x+1)dx = \frac{1}{2}\left((2x+1)\ln(2x+1) - (2x+1) + C\right).$$

(b) Notice that $\lim\limits_{t\to-\infty} e^t = 0$ and by the L'Hospital's Rule, we have

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}} = \lim_{t\to-\infty} -e^t = 0.$$

(c) By parts, we have that

$$\int xe^x dx = xe^x - e^x + C.$$

So,

$$\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty}\int_t^0 xe^x dx = \lim_{t\to-\infty}[xe^x - e^x]_t^0 = \lim_{t\to-\infty}[-1 - te^t + e^t] = -1$$

by using item (b).

Exercise 4:

(a) Let $f(x) = \sin(x)$. Then,

$$f'(x) = \cos(x), f''(x) = -\sin(x), f'''(x) = -\cos(x), f^{(4)}(x) = \sin(x), \dots.$$

So, $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{(4)}(0) = 0, \dots$. Since the derivatives repeat in a cycle of four, we have can write the Maclaurin series as follows:

$$\begin{aligned} f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \frac{f^{(4)}(0)}{4!} + \dots &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

(b) Let $a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^n \cdot (-1)\frac{x^{2n+1}\cdot x^2}{(2n+3)\cdot(2n+1)!}\cdot\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right| = \frac{x^2}{2n+3} \longrightarrow 0.$$

Therefore, $R = \infty$.

(c) Applying the L'Hospital's Rule five times, we have that

$$\lim_{x\to0}\frac{\sin(x) - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to0}\frac{\cos(x)}{5!} = \frac{1}{5!}.$$

(d) Notice first that

$$\sin(x) - x + \frac{1}{6}x^3 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) - x + \frac{1}{6}x^3 = \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots.$$

Then,

$$\frac{\sin(x) - x + \frac{1}{6}x^3}{x^5} = \frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \dots$$

and so $\lim_{x\to0}\frac{\sin(x) - x + \frac{1}{6}x^3}{x^5} = \frac{1}{5!}$.

Exercise 5:

(a) (F) For instance, $f(x) = x$ and $g(x) = x^2$. We have that

$$\int_1^2 x\cdot x^2 dx = \int_1^2 x^2 dx = \frac{15}{4}.$$

On the other hand,

$$\int_1^2 x dx = \frac{3}{2} \quad \text{and} \quad \int_1^2 x^2 dx = \frac{7}{3}.$$

So, $\left(\int_1^2 xdx\right)\left(\int_1^2 x^2dx\right) = \frac{3}{2}\cdot\frac{7}{3} = \frac{21}{6} \neq \frac{15}{4}$.

(b) (F) For instance, if $f(x) = 1$, then

$$\int x\cdot 1dx = \int xdx = \frac{x^2}{2} + C_1.$$

On the other hand, we have that

$$x\int dx = x\left(x + C_2\right) = x^2 + xC_2.$$

(c) (T) Let us notice that ax^2 and c are even functions and bx is odd.

(d) (F) For instance, let $f(x) = x - \frac{1}{2}$. Then,

$$\int_0^1 f(x)dx = \left[\frac{x^2}{2} - \frac{1}{2}x\right]_0^1 = 0$$

and $f(x) \neq 0$ on $(0, 1)$.

(e) (F) Consider, for instance, $f(x) = x^3$. We have that $f'(0) = 0$ but f does not have neither a local maximum nor a local minimum at 0.