Calculus 1 (BE5B01MA1)
Lab 2

Exercise 1. Find the limit of the function f when $x \to a$, where a is given. Justify your answer.

(a) $f(x) = 2x + 5$, where $a = -7$. $\lim_{x\to a} f(x) = -7$

(b) $f(x) = \dfrac{3}{\sqrt{3x+1}+1}$, where $a = 0$. $\lim\limits_{x\to a} f(x) = \dfrac{3}{2}$

(c) $f(x) = \dfrac{x-1}{\sqrt{x+3}-2}$, where $a = 1$. $\lim_{x\to a} f(x) = 4$

(d) $f(x) = \dfrac{3-\sqrt{x}}{9-x}$, where $a = 9$. $\lim\limits_{x\to a} f(x) = \dfrac{1}{6}$

(e) $f(x) = \dfrac{x^2+x}{x}$, where $a = 0$. $\lim_{x\to a} f(x) = 1$

Exercise 2. If the function f is defined in $\mathbb{R}$ and $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 1$, show that

$$\lim_{x\to 0} \frac{f(3x)}{x} = 3 \quad \text{and} \quad \lim_{x\to 0} \frac{f(x^2)}{x} = 0.$$

(Use the substitutions $u = 3x$ and $v = x^2$, respectively)

Definition. Let f be a function in $\mathbb{R}$.
(a). f is *bounded above* if there exists $K \in \mathbb{R}$ such that $f(x) \leqslant K$ for every $x \in D(f)$.
(b). f is *bounded below* if there exists $k \in \mathbb{R}$ such that $f(x) \geqslant k$ for every $x \in D(f)$.
(c). f is *bounded* if it is bounded above and bounded below.

Exercise 3. If $\lim\limits_{x\to a} f(x) = 0$ and g is a bounded function, show that

$$\lim_{x\to a} [f(x) \cdot g(x)] = 0.$$

(Use the Squeeze Theorem)

Exercise 4. Find $\lim\limits_{x\to 2^+} \sqrt{x-2}$ and check if the limit $\lim\limits_{x\to 0^-} \sqrt{x-2}$ exists.

Exercise 5. Find the left- and right-hand limits of the following functions.

(a) $\lim\limits_{x\to 0^+} \dfrac{x}{|x|}$. $(=1)$

(b) $\lim\limits_{x\to -1^+} \dfrac{x^2+3}{x^2-1}$. $(=-\infty)$

(c) $\lim\limits_{x\to 3^+} \dfrac{5}{x-3}$. $(=+\infty)$

(d) $\lim\limits_{x\to 0^+} \dfrac{2x+1}{x}$. $(=+\infty)$

(e) $\lim\limits_{x\to 0^-} \dfrac{x-3}{x^2}$. $(=-\infty)$

Exercise 6. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & , \text{ if } x \neq 1, \\ 3 & , \text{ if } x = 1. \end{cases}$$

(a) Calculate $\lim_{x\to 1} f(x)$.
(b) Decide if f is continuous at $x = 1$.

Exercise 7. Show that the equation $x^5 + x + 1 = 0$ has at least one root on the interval $[-1, 0]$. (Use the Intermediate Value Theorem)

Exercise 8. Find the following limits.

(a) $\lim\limits_{x\to +\infty} \dfrac{5x^3-6x+1}{6x^3+2}$. $\left(=\dfrac{5}{6}\right)$

(b) $\lim\limits_{x\to +\infty} \dfrac{\sqrt{x}+1}{x+3}$. $(=0)$

(c) $\lim\limits_{x\to -\infty} \dfrac{5-x}{3+2x}$. $\left(=-\dfrac{1}{2}\right)$

Exercise 9 (group discussion)**.** Let f be a **continuous** function defined in a neighborhood of the point $a = 1$ such that

$$f(x) = \frac{x^2-3x+2}{x-1} \quad \text{for} \quad x \neq 1.$$

Show that $f(1) = -1$ and justify your answer.

Undetermined forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad \infty^0, \quad 1^\infty \quad \text{and} \quad 0 \times \infty.$$

Operations with the symbols $\pm\infty$

(i) $\infty + \infty = \infty$.

(ii) $k \times \infty = \infty$ if $k > 0$.

(iii) $k^\infty = \infty$ if $k > 0$.

(iv) $\frac{k}{0} = \pm\infty$.

(v) $\infty \times \infty = \infty$.

(vi) $k \times \infty = -\infty$ if $k < 0$.

(vii) $\infty^p = \infty$ if $p > 0$.

(viii) $-\infty - \infty = -\infty$.

(ix) $(-\infty) \times \infty = -\infty$.

(x) $(-\infty) \times (-\infty) = \infty$.

(xi) $\infty^p = 0$ if $p < 0$.

(xii) $\infty \times (-\infty) = -\infty$.