CALCULUS 1 (BE5B01MA1)
LAB 5

Exercise 1. Suppose that $f(a) = g(a) = 0$, f', g' are continuous, and $g'(a) \neq 0$. Show the l'Hospital's Rule.

Exercise 2. Find the following limits.

(a) $\lim_{x \to \pi/2} \dfrac{\sec(x)}{1 + \tan(x)} = 1$

(b) $\lim_{x \to \infty} \dfrac{\ln(x)}{2\sqrt{x}} = 0$

(c) $\lim_{x \to \infty} \left(x \sin\left(\dfrac{1}{x}\right)\right) = 1$

(d) $\lim_{x \to 0} \left(\dfrac{1}{\sin(x)} - \dfrac{1}{x} \right) = 0$

(e) $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = 0$

(f) $\lim_{x \to 0^+} x^x = 1$

(g) $\lim_{x \to \infty} x^{1/x} = 1$

(h) $\lim_{x \to \pi/2} \left(\dfrac{\pi}{2} - y\right) \tan(y) = 1$

(i) $\lim_{x \to 0^+} \sqrt{x} \ln(x) = 0$

Exercise 3. Honza and Petr have found the limit of the function

$$f(x) = \frac{x - 3}{x^2 - 3} \quad \text{when} \quad x \to 3.$$

Nevertheless, their answers are different from each other. Decide which one is correct and **justify your answer**.

$$\textit{Honza Solution:} \quad \lim_{x \to 3} \frac{x - 3}{x^2 - 3} \overset{L'Hospital}{=} \lim_{x \to 3} \frac{1}{2x} = \frac{1}{6}.$$

$$\textit{Petr Solution:} \quad \lim_{x \to 3} \frac{x - 3}{x^2 - 3} = \frac{0}{6} = 0.$$

Exercise 4. Construct differentiable functions f and g such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ in such a way they satisfy:

(a) $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 3$ (b) $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = \infty$ (c) $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 0$

Exercise 5. Consider the following two limits and answer the questions.

$$\lim_{x \to \infty} \frac{\sqrt{9x + 1}}{x + 1} \quad \text{and} \quad \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{\sin(x)}}.$$

(a) They satisfy L'Hospital's Rule hypothesis?
(b) If the answer for (a) is yes, what happens if we try to apply that rule for these limits?
(c) Find the limits.

Exercise 6. Find $(f^{-1})'(a)$ when

(a) $f(x) = 2x^3 + 3x^2 + 7x + 4$ and $a = 4$.

(b) $g(x) = x^3 + 3\sin(x) + 2\cos(x)$ and $a = 2$.

(c) $h(x) = 3 + x^2 + \tan(\pi x/2)$, $-1 < x < 1$ and $a = 3$.

(d) $k(x) = \sqrt{x^3 + x^2 + x + 1}$ and $a = 2$.

Exercise 7. Let $f(x) = x + \ln(x)$ for $x > 0$.

(a) Show that f admits an inverse g and it is a differentiable function.

(b) Show that $g'(x) = \dfrac{g(x)}{1 + g(x)}$.

(c) Find $g(1)$, $g'(1)$, and $g''(1)$.

Exercise 8. Each of the following functions define y implicitly as a function of x. Find $\dfrac{dy}{dx}$.

(a) $y^3 = x + y$

(b) $y^3 + 2xy = \sqrt{x}$

(c) $\sqrt{x+y} = \sqrt{y+1}$

(d) $4\cos(x)\sin(y) = 1$

(e) $xy = \text{cotan}(xy)$

(f) $\sqrt{xy} = 1 + x^2 y$

Solutions: (a) $\dfrac{dy}{dx} = \dfrac{1}{3y^2 - 1}$, (b) $\dfrac{dy}{dx} = \dfrac{1 - 4\sqrt{xy}}{2\sqrt{x}(3y^2 + 2x)}$, (c) $\dfrac{dy}{dx} = \dfrac{\sqrt{1+y}}{\sqrt{x+y} - \sqrt{1+y}}$,

(d) $\dfrac{dy}{dx} = \tan(x)\tan(y)$, (e) $\dfrac{dy}{dx} = \dfrac{-y}{x}$, and (f) $\dfrac{dy}{dx} = \dfrac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$.

Exercise 9. Suppose that the equation

$$\frac{y}{x-y} - \frac{x}{y} + \sqrt{x} = 0$$

define implicitly y as a function of x nearby the point 1. Find $y'(1)$.

Exercise 10 (Group discussion)**.** Consider the following functions.

$$f(x) = \begin{cases} x + 2\,, & \text{if } x \neq 0, \\ 0 \quad, & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x + 1\,, & \text{if } x \neq 0, \\ 0 \quad, & \text{if } x = 0. \end{cases}$$

(a) Draw the graphs of f and g.

(b) Show that $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = 2$.

(c) Show that $\displaystyle\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1$.

(d) This contradicts the L'Hospital's Rule?