Calculus 1 (BE5B01MA1)
Midterm 1 (Week 6. 2019, Oct, 30th)

⋆ *This test has* **4 exercises** *with a total of* **12 points**.
⋆ *The maximum grade is* **12 points**.

Exercise 1. [**3pts**] Consider the function f defined as follows

$$f(x) = \begin{cases} -x & , \text{ if } x < 0, \\ 3 - x & , \text{ if } 0 \leqslant x < 3, \\ (x-3)^2 & , \text{ if } x > 3. \end{cases}$$

(a) **(0.6pts)** Find the domain of f.

(b) **(1.2pts)** Find each limit, if it exists, **justifying your answers.**

(i) $\lim_{x\to 0^+} f(x)$. (ii) $\lim_{x\to 0^-} f(x)$. (iii) $\lim_{x\to 0} f(x)$.

(iv) $\lim_{x\to 3^+} f(x)$. (v) $\lim_{x\to 3^-} f(x)$. (vi) $\lim_{x\to 3} f(x)$.

(c) **(0.3pts)** Where is f continuous?

(d) **(0.3pts)** Where is f discontinuous?

(e) **(0.6pts)** Sketch the graph of f.

Exercise 2. [**2pts**] Consider the following function:

$$h(x) = \frac{e^x - 1 - x - x^2/2}{x^3}.$$

(a) **(1pt)** Show that $f(x) = e^x - 1 - x - \frac{x^2}{2}$ and $g(x) = x^3$ satisfy the L'Hospital's Rule.

(b) **(1pt)** Show that $\lim_{x\to 0} h(x) = \frac{1}{6}$.

Exercise 3. [**3pts**] Decide if the following statements are **true** or **false**.

- If it is **true**, explain why.
- If it is **false**, give an example that disproves the statement.

(a) **(0.6pts)** If $f'(c) = 0$, then f has a local maximum or minimum at c.

(b) **(0.6pts)** If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1, 6)$.

(c) **(0.6pts)** If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.

(d) **(0.6pts)** If f is even, then f' is even.

(e) **(0.6pts)** If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$, then f is continuous at a.

Exercise 4. [**4pts**] Suppose that a function f satisfies the following conditions.

(i) **(0.5pts)** $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = +\infty$.

(ii) **(0.5pts)** $\lim_{x\to 6^+} f(x) = \lim_{x\to 6^-} f(x) = -\infty$.

(iii) **(0.6pts)** $f(0) = f'(-2) = f'(1) = f'(9) = 0$.

(iv) **(0.6pts)** $f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, +\infty)$.

(v) **(0.6pts)** $f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$.

(vi) **(0.6pts)** $f''(x) > 0$ on $(-\infty, 0)$ and $(12, +\infty)$.

(vii) **(0.6pts)** $f''(x) < 0$ on $(0, 6)$ and $(6, 12)$.

Sketch a possible graph of f.

Exercise 1.

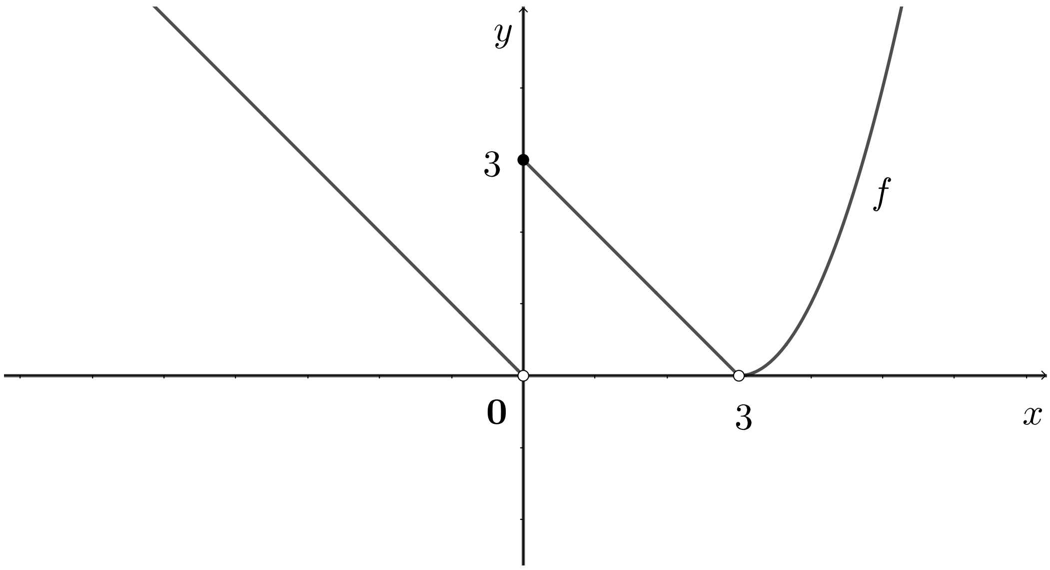
(a) $D(f) = \mathbb{R} \setminus \{3\}$.

(b) (i) 3.
(ii) 0.
(iii) it does not exist.
(iv) 0.
(v) 0.
(vi) 0.

(c) f is continuous on $\mathbb{R} \setminus \{0, 3\}$.

(d) f is discontinuous at $x = 0$ and $x = 3$.

(e)



Exercise 2.

(a) Since exponential and polynomials are differentiable functions, so are f and g. Notice that $g'(x) = 3x^2 \neq 0$ near 0 and also that we have an indeterminate limit of the form $\frac{0}{0}$. Therefore, we can apply the L'Hospital's Rule.

(b) We apply L'Hospital's Rule three times (note that all the hypothesis are always satisfied) to get

$$\begin{aligned} \lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3} &\overset{L'Hospital}{=} \lim_{x\to 0} \frac{e^x - 1 - x}{3x^2} \\ &\overset{L'Hospital}{=} \lim_{x\to 0} \frac{e^x - 1}{6x} \\ &\overset{L'Hospital}{=} \lim_{x\to 0} \frac{e^x}{6} = \frac{e^0}{6} = \frac{1}{6}. \end{aligned}$$

Exercise 3.

(a) **F**. The function $f(x) = x^3$ is such that $f'(0) = 0$ but it has no extreme values.

(b) **V**. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

(c) **F**. The functions $f(x) = 1$ and $g(x) = 2$ are such that $f'(x) = g'(x) = 0$ but $f(x) \neq g(x)$ for every $x \in \mathbb{R}$.

(d) **F**. The function $f(x) = x^2$ is even and $f'(x) = 2x$ is not.

(e) **F**. The function f from Exercise 1 is such that $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = 0$ but f is not defined at $x = 3$. Then, f cannot be continuous at $x = 3$.

Exercise 4.