On the Bishop-Phelps-Bollobás type theorems

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POSTECH

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Motivation & Historical background

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We say that a linear functional $x^* \in X^*$ **attains its norm** if there exists $x_0 \in S_X$ such that $|x^*(x_0)| = \|x^*\|$.

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$$|x_0^*(\alpha_n)| = \left|\sum_{n=1}^{\infty} \frac{1}{2^n}\alpha_n\right| \leq \sum_{n=1}^{\infty} \frac{1}{2^n}|\alpha_n| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 = \|x_0^*\|.$$

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So x_0^* is norm attaining.

Motivation & Historical background

James theorem

A Banach space X is reflexive if and only if every bounded linear functional attains its norm.

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$\mathrm{NA}(X,Y) \neq \emptyset$.

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BY USING HAHN-BANACH THEOREM...

$\mathrm{NA}(X,Y) \neq \emptyset$.

COUNTEREXAMPLE: $\dim(X) = \infty \Rightarrow \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$.

Motivation & Historical background

An important question

Question Is it true that, for every infinite-dimensional Banach space X, the set $\mathrm{NA}(X)$ contains a 2-dimensional subspace?

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Question Is it true that, for every infinite-dimensional Banach space X, the set $\mathrm{NA}(X)$ contains a 2-dimensional subspace?

Answer: No! (2017, M. Rmoutil)

Motivation & Historical background

The Bishop-Phelps-Bollobás theorem

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Bishop-Phelps theorem

Every element in X^* can be approximated by a norm attaining linear functional.

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In other words, $\overline{\mathrm{NA}(X)} = X^*$.

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Question (Bishop-Phelps)

Is it true for **bounded linear operators**?

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LINDENSTRAUSS' COUNTEREXAMPLE: There is a Banach space X such that

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Lindenstrauss studied **conditions** that the Banach spaces X and Y must satisfy to get that $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$.

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showing that the Bishop-Phelps result **does not hold** for bounded linear operators in general. On the other hand,

Lindenstrauss studied **conditions** that the Banach spaces X and Y must satisfy to get that $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$. For example, if X is reflexive then

$$\overline{NA(X,Y)} = \mathcal{L}(X,Y), \quad \forall\ Y.$$

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An important question

Question Is it true that every compact operator can be approximated by a compact norm attaining operator?

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Question Is it true that every compact operator can be approximated by a compact norm attaining operator?

Answer: No! (2014, M. Martín)

Motivation & Historical background

Bishop-Phelps-Bollobás theorem

Let X be a Banach space and $\varepsilon \in (0,1)$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{4},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$y^*(y) = 1 = \|y^*\|, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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- Bishop-Phelps-Bollobás $\Rightarrow$ Bishop-Phelps.

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- Bishop-Phelps-Bollobás $\Rightarrow$ Bishop-Phelps.
- It is **not** expected that there exists a Bishop-Phelps-Bollobás theorem version for bounded linear operators.

The Bishop-Phelps-Bollobás property

The Bishop-Phelps-Bollobás property (BPBp)

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A pair of Banach spaces (X, Y) is said to have the **BPBp** if for every $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there are $S \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ such that

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- (X, Y) has the BPBp $\Rightarrow \overline{\mathrm{NA}(X, Y)} = \mathcal{L}(X, Y)$.
- $(X, \mathbb{K})$ has the BPBp for every Banach space X. (Bishop-Phelps-Bollobás theorem)

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- (c_0, ℓ_∞).

Pairs of classic Banach spaces having the BPBp

DOMAIN SPACES \ RANGE SPACES	FD	ℓ_1^n	ℓ_p^n	ℓ_q^n	ℓ_∞^n	c_0	ℓ_1	ℓ_p	ℓ_q	ℓ_∞
FD	✓	✓	✓	✓	✓	✓				✓
ℓ_1^n	✓	✓	✓	✓	✓	✓				✓
ℓ_p^n	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_q^n	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_∞^n	✓	✓	✓	✓	✓	✓				✓
c_0						✓				✓
ℓ_1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_p	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_q	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_∞						✓				✓
$L_1(\mu)$						✓	$✓_\sigma$			✓
$L_1(\nu)$						✓	$✓_\sigma$			✓
$L_p(\mu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$L_p(\nu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$L_q(\mu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$L_q(\nu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$L_\infty(\mu)$						✓				✓
$L_\infty(\nu)$						✓				✓
$C(K)$			$✓_{\mathbb{R}}$	$✓_{\mathbb{R}}$		✓		$✓_{\mathbb{R}}$	$✓_{\mathbb{R}}$	✓
$C(S)$			$✓_{\mathbb{R}}$	$✓_{\mathbb{R}}$		✓		$✓_{\mathbb{R}}$	$✓_{\mathbb{R}}$	✓
$C_0(L_1)$						✓				✓
$C_0(L_2)$						✓				✓

Pairs of classic Banach spaces having the BPBp

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FD						
ℓ_1^n						
ℓ_p^n	✓	✓	✓	✓	✓	✓
ℓ_q^n	✓	✓	✓	✓	✓	✓
ℓ_∞^n						
c_0			✓	✓	✓	✓
ℓ_1	✓	✓	✓	✓	✓	✓
ℓ_p	✓	✓	✓	✓	✓	✓
ℓ_q	✓	✓	✓	✓	✓	✓
ℓ_∞						
$L_1(\mu)$	✓	✓	✓	✓	✓	✓
$L_1(\nu)$	✓	✓	✓	✓	✓	✓
$L_p(\nu)$	✓	✓	✓	✓	✓	✓
$L_p(\mu)$	✓	✓	✓	✓	✓	✓
$L_q(\mu)$	✓	✓	✓	✓	✓	✓
$L_q(\nu)$	✓	✓	✓	✓	✓	✓
$L_\infty(\mu)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	✓	✓	✓	✓
$L_\infty(\nu)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	✓	✓	✓	✓
$C(K)$			$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$
$C(S)$			$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$
$C_0(L_1)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$
$C_0(L_2)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$

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DOMAIN SPACES	RANGE SPACES: $L_\infty(\mu)$	$L_\infty(\nu)$	$C(K)$	$C(S)$	$C_0(L_1)$	$C_0(L_2)$
FD	✓$_\sigma$	✓$_\sigma$				
ℓ_1^n						
ℓ_p^n	✓	✓	✓	✓	✓	✓
ℓ_q^n	✓	✓	✓	✓	✓	✓
ℓ_∞^n						
c_0	✓$_\sigma$	✓$_\sigma$			✓	✓
ℓ_1			✓	✓		
ℓ_p	✓	✓	✓	✓	✓	✓
ℓ_q	✓	✓	✓	✓	✓	✓
ℓ_∞						
$L_1(\mu)$	✓$_\bullet$	✓$_\bullet$				
$L_1(\nu)$	✓$_\bullet$	✓$_\bullet$				
$L_p(\nu)$	✓	✓	✓	✓	✓	✓
$L_p(\mu)$	✓	✓	✓	✓	✓	✓
$L_q(\mu)$	✓	✓	✓	✓	✓	✓
$L_q(\nu)$	✓	✓	✓	✓	✓	✓
$L_\infty(\mu)$						
$L_\infty(\nu)$						
$C(K)$			✓$_\mathbb{R}$	✓$_\mathbb{R}$		
$C(S)$			✓$_\mathbb{R}$	✓$_\mathbb{R}$		
$C_0(L_1)$					✓$_\circ$	✓$_\circ$
$C_0(L_2)$					✓$_\circ$	✓$_\circ$

The Bishop-Phelps-Bollobás property

On the other hand,

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REMARKABLE EXAMPLE
(2015, R. Aron, Y. S. Choi, S. K. Kim, H. J. Lee, M. Martín)

The Bishop-Phelps-Bollobás property

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There is a Banach space $\mathcal{Y}$ such that

- $(\ell_1^2, \mathcal{Y})$ **fails** the BPBp but
- $\mathrm{NA}(\ell_1^2, Y) = \mathcal{L}(\ell_1^2, Y)$ for all Y.

Bishop-Phelps-Bollobás point property

The Bishop-Phelps-Bollobás point property (BPBpp)

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A pair (X, Y) is said to have the **BPBpp** if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there exists $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ such that

$$\|S(x_0)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

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$$\text{BPBpp} \Rightarrow \text{BPBp}.$$

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Uniformly smooth spaces

A Banach space X is said to **uniformly smooth** if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + th\| - 1}{t}$$

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- $L_p(\mu)$ is uniformly smooth for $1 < p < \infty$.

The Bishop-Phelps-Bollobás point property

(2016, D., S. K. Kim, H. J. Lee)

The Banach space X is uniformly smooth if and only if the pair $(X, \mathbb{K})$ has the BPBpp.

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Let X be a Banach space. Suppose that there is some Banach space Y such that the pair (X, Y) has the BPBpp. Then X is uniformly smooth.

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(Lindenstrauss) Property β

A Banach space Y is said to have **property β** with constant $0 \leq \rho < 1$ if there are sets $\{y_i : i \in I\} \subset S_Y$ and $\{y_i^* : i \in I\} \subset S_{Y^*}$ such that

(a) $y_i^*(y_i) = 1$ for all $i \in \Lambda$

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Typical examples: c_0 and ℓ_∞ have property β.

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Assume that X is uniformly smooth and that Y has the property β. Then the pair (X, Y) has the BPBpp.

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$$R(x) = y \quad \text{and} \quad \|R - \mathrm{Id}_H\| = \|x - y\|.$$

Quasi-transitive spaces with a control function

Banach spaces which are quasi-transitive with a control function

Quasi-transitive spaces with a control function

Quasi-transitivity

A Banach space X is **quasi-transitive** with a control function if there is $\beta : (0,2) \longrightarrow \mathbb{R}^+$ such that for every $\varepsilon \in (0,2)$, if $x, y \in S_X$ satisfy

$$\|x - y\| < \beta(\varepsilon),$$

there is $R \in \mathcal{L}(X,X)$ with $\|S\| = 1$ such that

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Example: Hilbert spaces are quasi-transitive with a control function.

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Natural question

- L_p-spaces are quasi-transitive with a control function?

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L_p-spaces are **not** quasi-transitive with a control function unless $p = 2$ or it is one-dimensional.

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- We do not know other examples of Banach spaces which are quasi-transitive with a control function besides Hilbert spaces.

Quasi-transitive spaces with a control function

Natural question

- We do not know other examples of Banach spaces which are quasi-transitive with a control function besides Hilbert spaces.
- Are Hilbert spaces the only one with this property?

Quasi-transitive spaces with a control function

(2017, D., Kadets, Kim, Lee, Martín)

Suppose that X is quasi-transitive with a control function and that (X, Y) has the BPBp. Then (X, Y) has the BPBpp.

Thank you
for your attention