

Some Bishop-Phelps-Bollobás type properties

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Notation

X, Y and Z are real or complex Banach spaces.

- \mathbb{K} is the field \mathbb{R} or \mathbb{C} ,
- B_X is the closed unit ball of X ,
- S_X is the unit sphere of X ,
- $\mathcal{L}(X, Y)$ continuous linear operators from X into Y ,
- $\text{NA}(X, Y)$ norm attaining operators from X into Y ,
- $X^* = \mathcal{L}(X, \mathbb{K})$ topological dual of X .
- $\text{NA}(X) = \text{NA}(X, \mathbb{K})$ norm attaining on X functionals.

Motivation & Historical background

Norm attaining functional

We say that a linear functional $x^* \in X^*$ **attains its norm** if there exists $x_0 \in S_X$ such that $|x^*(x_0)| = \|x^*\|$.

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Every element in X^* can be approximated by a norm attaining linear functional. In other words, $\overline{\text{NA}(X)} = X^*$.

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(1963, Lindenstrauss) Counterexample

There exists a strictly convex Banach space \mathcal{Z} such that the set $\text{NA}(c_0, \mathcal{Z})$ is **not** dense in $\mathcal{L}(c_0, \mathcal{Z})$.

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Bishop-Phelps-Bollobás theorem (Bollobás, 1970)

Let $\varepsilon \in (0, 2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$\operatorname{Re} x^*(x) > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = |y^*(y)| = 1, \quad \|y - x\| < \varepsilon, \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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Clearly, Bollobás' theorem \Rightarrow Bishop-Phelps theorem.

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 - Y is uniformly convex.
 - $Y = C(K)$ for K a compact Hausdorff space.

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- $(C(K), L_1(\mu))$ has the **BPBp**.
(2016, M. Acosta)

Pairs of classic Banach spaces having the BPBp

		RANGE SPACES									
		FD	ℓ_1^n	ℓ_p^n	ℓ_q^n	ℓ_∞^n	c_0	ℓ_1	ℓ_p	ℓ_q	ℓ_∞
DOMAIN SPACES	FD	✓	✓	✓	✓	✓	✓				✓
	ℓ_1^n	✓	✓	✓	✓	✓	✓				✓
	ℓ_p^n	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	ℓ_q^n	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	ℓ_∞^n	✓	✓	✓	✓	✓	✓				✓
	c_0						✓				✓
	ℓ_1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	ℓ_p	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	ℓ_q	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	ℓ_∞						✓				✓
	$L_1(\mu)$						✓	\checkmark_σ			✓
	$L_1(\nu)$						✓	\checkmark_σ			✓
	$L_p(\mu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	$L_p(\nu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	$L_q(\mu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	$L_q(\nu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	$L_\infty(\mu)$						✓				✓
	$L_\infty(\nu)$						✓				✓
	$C(K)$			✓ _R	✓ _R		✓		✓ _R	✓ _R	✓
	$C(S)$			✓ _R	✓ _R		✓		✓ _R	✓ _R	✓
$C_0(L_1)$						✓				✓	
$C_0(L_2)$						✓				✓	

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		$L_1(\mu)$	$L_1(\nu)$	$L_p(\mu)$	$L_p(\nu)$	$L_q(\mu)$	$L_q(\nu)$
D O M A I N S P A C E S	FD						
	ℓ_1^n						
	ℓ_p^n	✓	✓	✓	✓	✓	✓
	ℓ_q^n	✓	✓	✓	✓	✓	✓
	ℓ_∞^n						
	c_0			✓	✓	✓	✓
	ℓ_1	✓	✓	✓	✓	✓	✓
	ℓ_p	✓	✓	✓	✓	✓	✓
	ℓ_q	✓	✓	✓	✓	✓	✓
	ℓ_∞						
	$L_1(\mu)$	✓	✓	✓	✓	✓	✓
	$L_1(\nu)$	✓	✓	✓	✓	✓	✓
	$L_p(\nu)$	✓	✓	✓	✓	✓	✓
	$L_p(\mu)$	✓	✓	✓	✓	✓	✓
	$L_q(\mu)$	✓	✓	✓	✓	✓	✓
	$L_q(\nu)$	✓	✓	✓	✓	✓	✓
	$L_\infty(\mu)$	✓ _C	✓ _C	✓	✓	✓	✓
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D O M A I N S P A C E S	FD	\checkmark_σ	\checkmark_σ				
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	c_0	\checkmark_σ	\checkmark_σ			\checkmark	\checkmark
	ℓ_1			\checkmark	\checkmark		
	ℓ_p	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
	ℓ_q	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
	ℓ_∞						
	$L_1(\mu)$	\checkmark_\bullet	\checkmark_\bullet				
	$L_1(\nu)$	\checkmark_\bullet	\checkmark_\bullet				
	$L_p(\nu)$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
	$L_p(\mu)$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
	$L_q(\mu)$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
	$L_q(\nu)$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
	$L_\infty(\mu)$						
	$L_\infty(\nu)$						
	$C(K)$			$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$		
	$C(S)$			$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$		
	$C_0(L_1)$					\checkmark_\circ	\checkmark_\circ
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Bishop-Phelps-Bollobás **point** property (**BPBpp**)

A pair of Banach spaces (X, Y) is said to have the **BPBpp** if for every $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

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Clearly, the **BPBpp** implies the **BPBp**.

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- (H, Y) has the **BPBpp** for all Hilbert spaces H and any Y .
- There exists Y_0 such that $(L_p(\mu), Y_0)$ **fails** the **BPBpp**.

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- (X, \mathbb{K}) has the **BPBop** if and only if (X^*, \mathbb{K}) has the **BPBpp**.

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- (X, \mathbb{K}) has the **BPBop** if and only if X is uniformly convex.
(2014, S.K. Kim, H.J. Lee)
- (X, \mathbb{K}) has the **BPBop** if and only if (X^*, \mathbb{K}) has the **BPBpp**.
- If $\dim(X), \dim(Y) \geq 2$, then (X, Y) **fails** the **BPBop**.
(2018, D., V. Kadets, S.K. Kim, H.J. Lee, M. Martín)

The Bishop-Phelps-Bollobás operator property

(2017, D.)

(2018, J. Talponen)

(2018, D. Sain)

The Bishop-Phelps-Bollobás operator property

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“Local **BPBop**”

A pair of Banach spaces (X, Y) is said to have the **local BPBop** if for every $\varepsilon \in (0, 1)$ and $T \in S_{\mathcal{L}(X, Y)}$, there exists $\eta(\varepsilon, T) > 0$ such that if $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon, T),$$

there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

The Bishop-Phelps-Bollobás operator property

- If X is finite dimensional, then (X, Y) has the local **BPBop** for every Banach space Y .
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- If X is a reflexive space with the Kadec-Klee property, then (X, Y) has the local **BPBop** for compact operators, for every Banach space Y . (2018, D. Sain)

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- If X is finite dimensional, then (X, Y) has the local **BPBop** for every Banach space Y .
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- If X is a reflexive space with the Kadec-Klee property, then (X, Y) has the local **BPBop** for compact operators, for every Banach space Y . (2018, D. Sain)
- It was characterized in the setting with strictly convex domain spaces and compact operators.
(2018, J. Talponen)

The $L_{p,p}$ and the $L_{o,o}$

The $L_{p,p}$ and the $L_{o,o}$

(a) A pair (X, Y) has the $L_{p,p}$ if given $\varepsilon > 0$ and $x \in S_X$, there is $\eta(\varepsilon, x) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ satisfies

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The $L_{p,p}$ and the $L_{o,o}$

(b) A pair (X, Y) has the $L_{o,o}$ if given $\varepsilon > 0$ and $T \in S_{\mathcal{L}(X,Y)}$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies

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Remark: If X is reflexive, then (X, \mathbb{K}) has the $L_{p,p}$ if and only if (X^*, \mathbb{K}) has the $L_{o,o}$.

The $L_{p,p}$ and the $L_{0,0}$

To recall...

- If the one-side limit $\lim_{t \rightarrow 0^+} \frac{\|x+th\| - \|x\|}{t}$ exists uniformly for $h \in B_X$, we say that the norm of X is **SSD** at x .

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Then, X is Fréchet differentiable at x if and only if it is Gâteaux differentiable and **SSD** at x .

The $L_{p,p}$ and the $L_{o,o}$

(1995, G. Godefroy, V. Montesinos, V. Zizler)

(1993, C. Franchetti, R. Payá)

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Theorem

(a) (X, \mathbb{K}) has the $L_{p,p}$ if and only if the norm of X is **SSD**.

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- (a) (X, \mathbb{K}) has the $L_{p,p}$ if and only if the norm of X is **SSD**.
- (b) (X, \mathbb{K}) has the $L_{o,o}$ if and only if X is reflexive and the norm of X^* is **SSD**.

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Theorem

Suppose that the pair (X, \mathbb{K}) has the $L_{p,p}$. The norm of X is Gâteaux differentiable if and only if it is Fréchet differentiable.

The $L_{p,p}$ and the $L_{o,o}$

Definitions

- (a) X is **LUR** if for all $x, x_n \in S_X$ satisfying $\lim_n \|x_n + x\| = 2$, we have that $\lim_n \|x_n - x\| = 0$.

The $L_{p,p}$ and the $L_{o,o}$

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- (c) X is **MLUR** if whenever $(x_n), (y_n) \subset S_X$ are norm-one sequences in S_X with $\frac{1}{2}(x_n + y_n)$ converging to some $x_0 \in S_X$, we have that $\|x_n - y_n\| \rightarrow 0$.

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LUR \Rightarrow **w-LUR** \Rightarrow **strict convexity**

LUR \Rightarrow **MLUR** \Rightarrow **strict convexity**

The $L_{p,p}$ and the $L_{o,o}$

Theorem

Suppose that the pair (X, \mathbb{K}) has the $L_{o,o}$.

- (a) X is strictly convex if and only if X is MLUR.
- (b) X is strictly convex if and only if X^* is Fréchet differentiable.
(2018, J. Talponen)
- (c) X is w -LUR if and only if it is LUR.

The $L_{p,p}$ and the $L_{o,o}$

- X has the **Kadec-Klee property** if weak and norm topologies coincide on S_X .
- X^* has the **w^* -Kadec-Klee property** if the weak* and norm topologies coincide in S_{X^*} .

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Proposition

If X^* has the w^* -Kadec-Klee property, then the norm of X is **SSD** or, equivalently, the pair (X, \mathbb{K}) has the $L_{p,p}$.

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Corollary (2018, D. Sain) If X is a reflexive space which satisfies the Kadec-Klee property, then the pair (X, \mathbb{K}) has the $L_{o,o}$.

The $L_{p,p}$ and the $L_{o,o}$

Examples

If X is

- c_0 ,
- the predual of Lorentz sequence space $d_*(w, 1)$,
- the space VMO (which is the predual of the Hardy space H^1),
- finite dimensional spaces ℓ_1^N, ℓ_∞^N when $N \geq 2$,

then (X, \mathbb{K}) **satisfies** the $L_{p,p}$ but it **fails** the **BPBpp**.

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Proof

Indeed, it is known that the Hardy space H^1 of analytic functions on the ball and the Lorentz spaces $L_{p,1}(\mu)$ are **non-reflexive** dual spaces that have the w^* -Kadec-Klee property.

The vector-value case

The $L_{p,p}$ and the $L_{o,o}$

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- If the pair (X, Y) has the $L_{p,p}$, then (X, \mathbb{K}) has the $L_{p,p}$.
- If Y has property β and (X, \mathbb{K}) has $L_{p,p}$, then so does (X, Y) .

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

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In particular, (c_0, c_0) has the $\mathbf{L}_{p,p}$.

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Remark: There is a Banach space X which is **SSD** but the pair (X, Y) **fails** the $L_{p,p}$ for some Y .

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Remark: There is a Banach space X which is **SSD** but the pair (X, Y) **fails** the $L_{p,p}$ for some Y .

Remark: (ℓ_1, Y) **fails** the $L_{p,p}$ for all Banach spaces Y , since the norm of ℓ_1 is **SSD** only at the points in the unit sphere which are sequences with finitely many nonzero terms.

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Let X be a uniformly convex Banach space. Then,

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Theorem

Let X be a uniformly convex Banach space. Then,

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(but they **fail** to have the **BPBpp**).

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there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

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The L_p and the L_o

(b) A pair (X, Y) has the L_o if given $\varepsilon > 0$ and $T \in S_{\mathcal{L}(X, Y)}$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies

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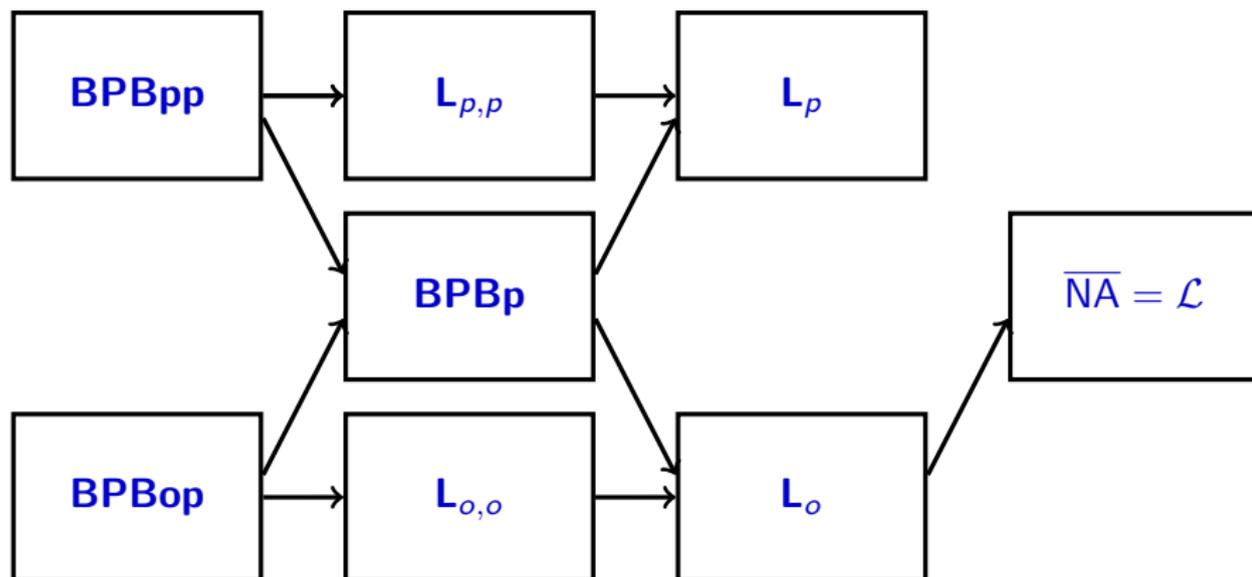
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$\overline{\text{NA}(X, \mathbb{R}^2)} = \mathcal{L}(X, \mathbb{R}^2)$ for all Banach spaces X ?

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For $1 < p < 2$, the pair $(L_p(\mu), Y)$ has the **BPBpp** for all Banach spaces Y ?

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A direct proof for finite dimensional Banach spaces X, Y such that the pair (X, Y) has the **BPBp**.

Question 3

For $1 < p < 2$, the pair $(L_p(\mu), Y)$ has the **BPBpp** for all Banach spaces Y ?

Question 4

We do not know whether the pair (ℓ_p^2, ℓ_q) has the $\mathbf{L}_{p,p}$ for $1 < p, q < \infty$.

Thank you
for your attention