Smooth norms on dense subspaces

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Work in progress with **Petr Hájek** and **Tommaso Russo**
Robert Deville's 60 birthday
June, 2019, Métabief (France)

- Smooth renormings of Banach spaces

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(ii) *If the dual norm is Fréchet differentiable, then* X *is reflexive.*

(iii) *If the dual norm on* X^* *is LUR, then* $\|\cdot\|$ *is Fréchet differentiable.*

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Question *Does every Asplund Banach space admit a C^1-smooth bump function?*

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- (J. Vanderwerff, 1992) *If X is a separable Banach space and L is a subspace of dimensional $\aleph_0$, then X admits an equivalent LUR norm which is Fréchet differentiable on $L\setminus\{0\}$. In particular, any* **normed** *space of dimension $\aleph_0$ admits a Fréchet differentiable norm.*

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Q3. What can one say about the whole space X if there exists a dense subspace Y which admits a C^k-smooth norm?

The results

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Given a normed space $(X, \|\cdot\|)$ and $\varepsilon > 0$, we say that a new norm $|||\cdot|||$ approximates the original one $\|\cdot\|$ if

$$|||x||| \leq \|x\| \leq (1+\varepsilon)|||x|||$$

for all $x \in X$.

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Then, the Minkowski functional μ *on* B *is an equivalent* C^k*-smooth norm on* X.

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Theorem 1: *The space ℓ_∞^F admits an analytic norm which approximates the original one.*

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Corollary 2: *Let X be a separable Banach space. Then, there is a dense subspace Y of X which admits an analytic norm and approximates the original one.*

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Corollary 2: *Let X be a separable Banach space. Then, there is a dense subspace Y of X which admits an analytic norm and approximates the original one.*

Corollary 3: *The normed space $\mathcal{F}$ of all finitely supported vectors in $\ell_1(c)$, where c denotes a set of cardinality continuum, endowed with the ℓ_1-norm, admits an equivalent analytic norm which approximates the original one.*

The results

Theorem 4: *Let* X *be a Banach space with a suppression* 1*-unconditional Schauder basis* $\{e_\gamma\}_{\gamma\in\Gamma}$ *and set* $Y := \mathrm{span}\{e_\gamma\}_{\gamma\in\Gamma}$. *Then,* Y *is a dense subspace of* X *which admits a* C^∞*-smooth norm and approximates the original one.*

Our problem

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Thank you
for your attention

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Q2. If a dense subspace Y admits a C^k-smooth norm, then X is Asplund?

Q3. Is there a Banach space X in which no dense subspace have a smooth norm? $\ell_\infty^{F,c}(\Gamma)$