On some dense subspaces with C^k-smooth norms

Sheldon Dantas

Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Mathematics

Research supported by the project OPVVV CAAS CZ.02.1.01/0.0/0.0/16_019/0000778, Excelentní výzkum

Centrum pokročilých aplikovaných přírodních věd

(Center for Advanced Applied Science)

Work in progress with Petr Hájek and Tommaso Russo

August, 2019, Pohang (South Korea)

- Smooth renormings of Banach spaces

- Smooth renormings of Banach spaces
- **Renorming theory** vs. **Smooth functions**

- Smooth renormings of Banach spaces
- **Renorming theory** vs. **Smooth functions**
    - R. Deville, G. Godefroy, and V. Zizler
      Smoothness and Renorming in Banach spaces

- Smooth renormings of Banach spaces
- **Renorming theory** vs. **Smooth functions**
  - R. Deville, G. Godefroy, and V. Zizler
    Smoothness and Renorming in Banach spaces
  - P. Hájek and M. Johanis
    Smooth Analysis in Banach spaces

- Smooth renormings of Banach spaces
- **Renorming theory** vs. **Smooth functions**
  - R. Deville, G. Godefroy, and V. Zizler
    Smoothness and Renorming in Banach spaces
  - P. Hájek and M. Johanis
    Smooth Analysis in Banach spaces
  - G. Godefroy
    Renormings in Banach spaces

- Smooth renormings of Banach spaces
- **Renorming theory** vs. **Smooth functions**
  - R. Deville, G. Godefroy, and V. Zizler
    Smoothness and Renorming in Banach spaces
  - P. Hájek and M. Johanis
    Smooth Analysis in Banach spaces
  - G. Godefroy
    Renormings in Banach spaces
  - V. Zizler
    Nonseparable Banach spaces

Motivation

Motivation

Theorem: *Let* $(X, \|\cdot\|)$ *be a Banach space.*

Motivation

Theorem: *Let* $(X, \|\cdot\|)$ *be a Banach space.*

(i) $\|\cdot\|$ *is* C^1*-smooth whenever it is Fréchet differentiable.*

Motivation

Theorem: *Let* $(X, \|\cdot\|)$ *be a Banach space.*

(i) $\|\cdot\|$ *is* C^1*-smooth whenever it is Fréchet differentiable.*

(ii) *If the dual norm is Fréchet differentiable, then* X *is reflexive.*

Motivation

Theorem: *Let* $(X, \|\cdot\|)$ *be a Banach space.*

(i) $\|\cdot\|$ *is* C^1*-smooth whenever it is Fréchet differentiable.*

(ii) *If the dual norm is Fréchet differentiable, then* X *is reflexive.*

(iii) *If the dual norm on* X^* *is LUR, then* $\|\cdot\|$ *is Fréchet differentiable.*

Motivation

There is a stronger result:

Theorem (M. Fabian, 1987) *If a Banach space X admits a C^1-smooth bump, then it is Asplund.*

Motivation

There is a stronger result:

Theorem (M. Fabian, 1987) *If a Banach space X admits a C^1-smooth bump, then it is Asplund.*

Question *Does every Asplund Banach space admit a C^1-smooth bump function?*

Motivation

- (V.Z. Meshkov, 1978) *A Banach space X is isomorphic to a Hilbert space, whenever both X and X^* admit a C^2-smooth bump.*

Motivation

- (V.Z. Meshkov, 1978) *A Banach space X is isomorphic to a Hilbert space, whenever both X and X^* admit a C^2-smooth bump.*
- (M. Fabian, J.H.M. Whitfield, and V. Zizler, 1983) *If a Banach space X admits a C^2-smooth bump, then X contains an isomorphic copy of c_0 or X is superreflexive of type 2.*

Motivation

- (V.Z. Meshkov, 1978) *A Banach space X is isomorphic to a Hilbert space, whenever both X and X^* admit a C^2-smooth bump.*
- (M. Fabian, J.H.M. Whitfield, and V. Zizler, 1983) *If a Banach space X admits a C^2-smooth bump, then X contains an isomorphic copy of c_0 or X is superreflexive of type 2.*
- (R. Deville, 1989) *The existence of a C^∞-smooth bump on a Banach space X that contain no copy of c_0 implies that X is of cotype $2k$, for some k, and it contain a copy of ℓ_{2k}.*

Motivation

- (V.Z. Meshkov, 1978) *A Banach space* X *is isomorphic to a Hilbert space, whenever both* X *and* X^* *admit a* C^2*-smooth bump.*
- (M. Fabian, J.H.M. Whitfield, and V. Zizler, 1983) *If a Banach space* X *admits a* C^2*-smooth bump, then* X *contains an isomorphic copy of* c_0 *or* X *is superreflexive of type* 2*.*
- (R. Deville, 1989) *The existence of a* C^∞*-smooth bump on a Banach space* X *that contain no copy of* c_0 *implies that* X *is of cotype* $2k$*, for some* k*, and it contain a copy of* ℓ_{2k}*.*
- (J. Vanderwerff, 1992) *If* X *is a separable Banach space and* L *is a subspace of dimensional* $\aleph_0$*, then* X *admits an equivalent LUR norm which is Fréchet differentiable on* $L \setminus \{0\}$*.*

Motivation

- (V.Z. Meshkov, 1978) *A Banach space X is isomorphic to a Hilbert space, whenever both X and X^* admit a C^2-smooth bump.*
- (M. Fabian, J.H.M. Whitfield, and V. Zizler, 1983) *If a Banach space X admits a C^2-smooth bump, then X contains an isomorphic copy of c_0 or X is superreflexive of type 2.*
- (R. Deville, 1989) *The existence of a C^∞-smooth bump on a Banach space X that contain no copy of c_0 implies that X is of cotype $2k$, for some k, and it contain a copy of ℓ_{2k}.*
- (J. Vanderwerff, 1992) *If X is a separable Banach space and L is a subspace of dimensional $\aleph_0$, then X admits an equivalent LUR norm which is Fréchet differentiable on $L \setminus \{0\}$. In particular, any* **normed** *space of dimension $\aleph_0$ admits a Fréchet differentiable norm.*

Our problem

Our problem

(A. Guirao, V. Montesinos, and V. Zizler, 2016)

Our problem

(A. Guirao, V. Montesinos, and V. Zizler, 2016)

Let Γ *be an uncountable set and* $\mathcal{F}$ *be a normed space of all finitely supported vectors in* $\ell_1(\Gamma)$ *endowed with the* ℓ_1*-norm. Does* $\mathcal{F}$ *admit a Fréchet smooth norm?*

Our problem

(A. Guirao, V. Montesinos, and V. Zizler, 2016)

Let Γ be an uncountable set and $\mathcal{F}$ be a normed space of all finitely supported vectors in $\ell_1(\Gamma)$ endowed with the ℓ_1-norm. Does $\mathcal{F}$ admit a Fréchet smooth norm?

Q1. If a dense subspace Y admits a C^k-smooth norm, then the whole space X also does?

Our problem

(A. Guirao, V. Montesinos, and V. Zizler, 2016)

Let Γ be an uncountable set and $\mathcal{F}$ be a normed space of all finitely supported vectors in $\ell_1(\Gamma)$ endowed with the ℓ_1-norm. Does $\mathcal{F}$ admit a Fréchet smooth norm?

Q1. If a dense subspace Y admits a C^k-smooth norm, then the whole space X also does?

Q2. If a dense subspace Y admits a C^k-smooth norm, then X is Asplund?

Our problem

(A. Guirao, V. Montesinos, and V. Zizler, 2016)

Let Γ be an uncountable set and $\mathcal{F}$ be a normed space of all finitely supported vectors in $\ell_1(\Gamma)$ endowed with the ℓ_1-norm. Does $\mathcal{F}$ admit a Fréchet smooth norm?

Q1. If a dense subspace Y admits a C^k-smooth norm, then the whole space X also does?

Q2. If a dense subspace Y admits a C^k-smooth norm, then X is Asplund?

Q3. What can one says about the whole space X if there exists a dense subspace Y which admits a C^k-smooth norm?

The results

The results

Given a normed space $(X, \|\cdot\|)$ and $\varepsilon > 0$, we say that a new norm $|||\cdot|||$ approximates the original one $\|\cdot\|$ if

$$|||x||| \leq \|x\| \leq (1+\varepsilon)|||x|||$$

for all $x \in X$.

The results

Implicit functional theorem for Minkowski functionals
(P. Hájek and M. Johanis, *Smooth Analysis in Banach spaces*)

The results

Implicit functional theorem for Minkowski functionals
(P. Hájek and M. Johanis, *Smooth Analysis in Banach spaces*)

Let $(X, \|\cdot\|)$ *be a normed space and* D *be a nonempty, open, convex, symmetric subset of* X*.*

The results

Implicit functional theorem for Minkowski functionals
(P. Hájek and M. Johanis, *Smooth Analysis in Banach spaces*)

Let $(X, \|\cdot\|)$ *be a normed space and* D *be a nonempty, open, convex, symmetric subset of* X. *Let* $f : D \longrightarrow \mathbb{R}$ *be even, convex, and continuous.*

The results

Implicit functional theorem for Minkowski functionals
(P. Hájek and M. Johanis, *Smooth Analysis in Banach spaces*)

Let $(X, \|\cdot\|)$ *be a normed space and* D *be a nonempty, open, convex, symmetric subset of* X. *Let* $f : D \longrightarrow \mathbb{R}$ *be even, convex, and continuous. Suppose that there is* $a > f(0)$ *such that the level set* $B := \{f \leq a\}$ *is bounded and closed in* X.

The results

Implicit functional theorem for Minkowski functionals
(P. Hájek and M. Johanis, *Smooth Analysis in Banach spaces*)

Let $(X, \|\cdot\|)$ *be a normed space and* D *be a nonempty, open, convex, symmetric subset of* X*. Let* $f : D \longrightarrow \mathbb{R}$ *be even, convex, and continuous. Suppose that there is* $a > f(0)$ *such that the level set* $B := \{f \leq a\}$ *is bounded and closed in* X*. Assume further that there is an open set* O *with* $\{f = a\} \subset O$ *such that* f *is* C^k*-smooth on* O*, where* $k \in \mathbb{N} \cup \{\infty, \omega\}$*.*

The results

Implicit functional theorem for Minkowski functionals
(P. Hájek and M. Johanis, *Smooth Analysis in Banach spaces*)

Let $(X, \|\cdot\|)$ *be a normed space and* D *be a nonempty, open, convex, symmetric subset of* X*. Let* $f : D \longrightarrow \mathbb{R}$ *be even, convex, and continuous. Suppose that there is* $a > f(0)$ *such that the level set* $B := \{f \leq a\}$ *is bounded and closed in* X*. Assume further that there is an open set* O *with* $\{f = a\} \subset O$ *such that* f *is* C^k*-smooth on* O*, where* $k \in \mathbb{N} \cup \{\infty, \omega\}$*.*

Then, the Minkowski functional μ *on* B *is an equivalent* C^k*-smooth norm on* X*.*

The results

Let ℓ_∞^F denote the dense linear subspace of ℓ_∞ consisting of finitely-valued sequences.

The results

Let ℓ_∞^F denote the dense linear subspace of ℓ_∞ consisting of finitely-valued sequences.

Theorem 1: *The space ℓ_∞^F admits an analytic norm which approximates the original one.*

The results

Two consequences:

The results

Two consequences:

Corollary 2: *Let X be a separable Banach space. Then, there is a dense subspace Y of X which admits an analytic norm and approximates the original one.*

The results

Two consequences:

Corollary 2: *Let X be a separable Banach space. Then, there is a dense subspace Y of X which admits an analytic norm and approximates the original one.*

Corollary 3: *The normed space $\mathcal{F}$ of all finitely supported vectors in $\ell_1(c)$, where c denotes a set of cardinality continuum, endowed with the ℓ_1-norm, admits an equivalent analytic norm which approximates the original one.*

The results

Theorem 4: *Let X be a Banach space with a suppression 1-unconditional Schauder basis $\{e_\gamma\}_{\gamma\in\Gamma}$ and set $Y := \operatorname{span}\{e_\gamma\}_{\gamma\in\Gamma}$. Then, Y is a dense subspace of X which admits a C^∞-smooth norm and approximates the original one.*

Our problem

Q1. If a dense subspace Y admits a C^k-smooth norm, then X is Asplund?

Our problem

Q1. If a dense subspace Y admits a C^k-smooth norm, then X is Asplund?

Q2. Is there a Banach space X in which no dense subspace have a smooth norm?

Our problem

Q1. If a dense subspace Y admits a C^k-smooth norm, then X is Asplund?

Q2. Is there a Banach space X in which no dense subspace have a smooth norm?

Our problem

GENERAL QUESTION

How different can two dense subspaces of a Banach space be?

Thank you
for your attention