Subdifferentiability and the Bishop-Phelps-Bollobás theorem

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Strong subdifferentiability of the norm

SSD

We say that the norm of a Banach space X is strongly subdifferentiable (**SSD**, for short) at a point $u \in S_X$ if the one-sided limit

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- Consider ϕ_n on B_X defined by

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- Consider ϕ_n on B_X defined by

$$\phi_n(x) = \frac{1}{n}\left(\left\|u + \frac{x}{n}\right\| - 1\right) = \|nu + x\| - n.$$

- Then, the norm of X is SSD iff $\{\phi_n\}$ converges uniformly on B_X.

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- A Banach space with an SSD norm is Asplund.
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Let $x^* \in S_{X^*}$ be such that

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By the Bishop-Phelps-Bollobás theorem, there are $(y, y^*) \in S_X \times S_{X^*}$ such that

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If $z^* \in S_{X^*}$ is such that $z^*(x) = 1$, since $\|y - x\| < \delta$ and $y^*(y) = 1$, we have $\|z^* - y^*\| < \delta$.

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Theorem (C. Franchetti and R. Payá, 1993)

The pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ if and only the norm of X is SSD.

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and whenever $x^* \in S_{X^*}$ satisfies $\|x^* - x_n^*\| < \varepsilon_0$, we have $|x^*(x_0)| < 1$. By the Banach-Alaoglu theorem, there is a subnet of (x_n^*) such that $x_n^* \xrightarrow{w^*} x_0^*$ for some $x_0^* \in B_{X^*}$.

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A dual Banach space X^* has the w*-Kadec-Klee property if $x_\alpha \xrightarrow{\|.\|} 0$ whenever $\|x_\alpha\| \to \|x\|$ and $x_\alpha \xrightarrow{w^*} x$.

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A dual Banach space X^* has the w^*-Kadec-Klee property if $x_\alpha \xrightarrow{\|.\|} 0$ whenever $\|x_\alpha\| \to \|x\|$ and $x_\alpha \xrightarrow{w^*} x$.

Theorem

If X^* has the w^*-Kadec-Klee property, then the norm of X is SSD.

Property $\mathbf{L}_{p,p}$ for operators

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Examples of linear operators

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- (X, Y) has the $\mathbf{L}_{p,p}$ for finite-dimensional spaces X, Y.

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- (X, Y) has the $\mathbf{L}_{p,p}$ for some $Y \Rightarrow X$ must be SSD.
- (X, Y) has the $\mathbf{L}_{p,p}$ for finite-dimensional spaces X, Y.
  - If Y is strictly convex and (ℓ_1^2, Y) has the $\mathbf{L}_{p,p}$, then Y must be uniformly convex.

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- (X, Y) has the $\mathbf{L}_{p,p}$ for finite-dimensional spaces X, Y.
- (ℓ_1^N, X) has the $\mathbf{L}_{p,p}$ when X is uniformly convex.

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Examples of linear operators

- (X, Y) has the $\mathbf{L}_{p,p}$ for some $Y \Rightarrow X$ must be SSD.
- (X, Y) has the $\mathbf{L}_{p,p}$ for finite-dimensional spaces X, Y.
- (ℓ_1^N, X) has the $\mathbf{L}_{p,p}$ when X is uniformly convex.
- $(c_0, L_p(\mu))$ has the $\mathbf{L}_{p,p}$ for μ positive measures and $1 \leq p < \infty$.

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We say that $(X, Y; Z)$ has the $\mathbf{L}_{p,p}$ if given $\varepsilon > 0$ and $(x, y) \in S_X \times S_Y$, there is $\eta(\varepsilon, (x, y)) > 0$ such that whenever $A \in \mathcal{L}(X, Y; Z)$ with $\|A\| = 1$ satisfies

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there is $B \in \mathcal{L}(X, Y; Z)$ with $\|B\| = 1$ such that

$$\|B(x, y)\| = 1 \quad \text{and} \quad \|B - A\| < \varepsilon.$$

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- If $(X, Y; Z)$ has the $\mathbf{L}_{p,p}$, then X and Y are both SSD.

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Consider $\hat{A} \in S_{(\ell_p \hat{\otimes}_\pi \ell_q)^*}$. Then,

$$|\hat{A}(x \otimes y)| = |A(x, y)| > 1 - \eta(\varepsilon, x \otimes y).$$

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- Then, $(\ell_p \hat{\otimes}_\pi \ell_q; \mathbb{K})$ has the $\mathbf{L}_{p,p}$ for $2 < p, q < \infty$.

Fix $\varepsilon > 0$ and $(x, y) \in S_{\ell_p} \times S_{\ell_q}$. Consider $\eta(\varepsilon, x \otimes y) > 0$. Let $A \in \mathcal{L}(\ell_p, \ell_q; \mathbb{K})$ with $\|A\| = 1$ with

$$|A(x, y)| > 1 - \eta(\varepsilon, x \otimes y).$$

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$$|\hat{A}(x \otimes y)| = |A(x, y)| > 1 - \eta(\varepsilon, x \otimes y).$$

Then, there is $\hat{B} \in S_{(\ell_p \hat{\otimes}_\pi \ell_q)^*}$ with

$$|B(x, y)| = |\hat{B}(x \otimes y)| = 1 \quad \text{and} \quad \|B - A\| = \|\hat{B} - \hat{A}\| < \varepsilon.$$

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SSD in projective tensor products

Theorem

(a). If $2 < p, q < \infty$, then $\ell_p \hat{\otimes}_\pi \ell_q$ is SSD.

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SSD in projective tensor products

Theorem

(a). If $2 < p, q < \infty$, then $\ell_p \hat{\otimes}_\pi \ell_q$ is SSD.

(b). If $2 < p, q < \infty$, then $(\ell_p, \ell_q; \mathbb{K})$ has the $\mathbf{L}_{p,p}$.

(c). If $p^{-1} + q^{-1} \geq 1$ or one of them is 1 or ∞, then $\ell_p \hat{\otimes}_\pi \ell_q$ is **not** SSD.

Questions

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Thank you
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