

# Microtransitivity in $L_p$ -spaces

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# Notation

- $X, Y$  are real or complex Banach spaces
- $\mathbb{K}$  is the field  $\mathbb{R}$  or  $\mathbb{C}$
- $B_X$  is the closed unit ball of  $X$
- $S_X$  is the unit sphere of  $X$
- $\mathcal{L}(X, Y)$  continuous linear operators from  $X$  into  $Y$ 
  - If  $X = Y$ , then  $\mathcal{L}(X, X) = \mathcal{L}(X)$
- $\mathcal{G}(X)$  all surjective linear isometries from  $X$  to  $X$

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*Is every transitive separable Banach space isometrically isomorphic to a Hilbert space?*

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- (i)  $ex = x, \forall x \in T,$
- (ii)  $g(hx) = (gh)x, \forall g, h \in G, \forall x \in T.$

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- (i) If the action is micro-transitive, then the orbits of the elements are open. (F.D. Ancel, 1987)
- (ii) So, they produce a partition of the space  $T$  as a disjoint union of open sets.
- (iii) Therefore, if  $T$  is connected, micro-transitivity  $\Rightarrow$  transitivity.

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Hilbert spaces have the above property with  $\beta(\varepsilon) = \varepsilon$ .

(see, for example, M. Acosta, M. Mastyło, and M. Soleimani-M., 2018)

# Notation

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## Definition (Uniform micro-semitransitivity)

We say that the norm of  $X$  is **uniformly micro-semitransitive** if there is a function  $\beta : (0, 2) \rightarrow \mathbb{R}^+$  such that whenever  $x, y \in S_X$  satisfies

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Indeed, we take  $x_1, \dots, x_n \in S_X$  such that

$$x_1 = x, \quad x_n = y \quad \text{and} \quad \|x_{i+1} - x_i\| < \beta \left( \frac{1}{2} \right), \forall i = 1, \dots, n-1.$$



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- (c) For every Banach space  $X$ , given  $x, y \in S_X$ , there is  $T \in \mathcal{L}(X)$  with  $\|T\| = 1$  such that  $Tx = y$ .
- (d) If  $X$  is uniformly micro-semitransitive and  $Y$  is a 1-complemented subspace of  $X$ , then  $Y$  is uniformly micro-semitransitive.

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(M. Acosta, R. Aron, D. García, and M. Maestre, 2008)

A pair  $(X, Y)$  of Banach spaces has the **BPBp** if for every  $\varepsilon > 0$ , there exists  $\eta(\varepsilon) > 0$  such that whenever  $T \in \mathcal{L}(X, Y)$  with  $\|T\| = 1$  and  $x_0 \in S_X$  satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there are  $S \in \mathcal{L}(X, Y)$  with  $\|S\| = 1$  and  $x \in S_X$  such that

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**(a) (Bollobás, 1963)**  $(X, \mathbb{K})$  has the **BPBp**, for all Banach  $X$ .

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**(a)** (Bollobás, 1963)  $(X, \mathbb{K})$  has the **BPBp**, for all Banach  $X$ .

**(b)** (S.K. Kim and H.J. Lee, 2014) If  $X$  is uniformly convex, then  $(X, Y)$  has the **BPBp** for all Banach  $Y$ .

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## Bishop-Phelps-Bollobás point property (D., S.K. Kim, and H.J. Lee (2016))

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(Transitive + superflexive  $\Rightarrow$  uniformly convex (Finet, 1986))

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If, moreover,  $X$  is isomorphic to a Hilbert space, then the above conditions are indeed equivalent to

- (e)  $\exists C > 0$  such that  $\delta_X(\varepsilon) \geq C \varepsilon^2$  ( $0 < \varepsilon < 2$ ), where the constant  $C$  depends only on the modulus of convexity of  $X$ , on the function  $\beta(\cdot)$  of the definition of uniform micro-transitivity, and on the Banach-Mazur distance from  $X$  to the Hilbert space.

## Corollary 5

If  $X$  is micro-transitive, then

- (a) For every Banach space  $Y$ , the pair  $(X, Y)$  has the **BPBpp**,
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**Remark:** For  $1 \leq p < \infty$ , there are (non-separable)  $L_p$ -spaces whose standard norms are transitive but **they are not** micro-transitive unless  $p = 2$ .

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Indeed, for  $p > 2$ , there is **no** isomorphism  $T \in \mathcal{L}(\ell_p^2)$  with  $\|T\| = 1$  such that

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Since  $\ell_p^2$  is always 1-complemented in  $L_p(\mu)$ , we are done.

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*Every space that is of type 2 and cotype 2 is isomorphic to a Hilbert space  
(S. Kwapien, 1972)*

Thank you  
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