On norm attaining theory and Bishop-Phelps-Bollobás properties

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Motivation and Historical background

Definition

A functional $x^* \in X^*$ **attains the norm** if there is $x_0 \in S_X$ such that
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James Theorem

A Banach space X is reflexive if and only if every functional in X^* attains the norm.

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When does the set $\mathrm{NA}(X)$ is norm dense in X^*?

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Is it true for bounded linear operators?

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$T \in \mathcal{L}(X, Y)$ **attains the norm** if there is $x_0 \in S_X$ such that

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Bishop-Phelps' question

$\overline{\mathrm{NA}(X, Y)} = \mathcal{L}(X, Y)$ for every X, Y?

Lindenstrauss counterexample (1963)

There is a Banach space X such that

$$\overline{\text{NA}(X, X)} \neq \mathcal{L}(X, X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators in general.

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- c_0 and ℓ_∞ have property B (property β).

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Examples (Lindenstrauss, 1963)

- $L_1[0,1]$ fails property A.
- Y strictly convex containing c_0 fails property B.

Examples

- Infinite dimensional $L_p(\mu)$ spaces with $1 < p < \infty$ fail property B.
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Question

Can **finite-rank** operators be approximated by norm attaining operators?

The Bollobás Theorem

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Bishop-Phelps-Bollobás theorem (1970)

Let $\varepsilon \in (0, 2)$. Given $(x, x^*) \in B_X \times B_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = |y^*(y)| = 1, \quad \|y - x\| < \varepsilon, \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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It is **not** expected that there exists a Bishop-Phelps-Bollobás theorem version for bounded linear operators in general.

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Bishop-Phelps-Bollobás property (BPBp)

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Aron, Choi, Kim, Lee, and Martín, 2015

There is a Banach space Y such that

- (ℓ_1^2, Y) fails the **BPBp** but
- $\mathrm{NA}(\ell_1^2, Y) = \mathcal{L}(\ell_1^2, Y)$.

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(b) $X = \ell_1$ and
  - $Y = L_1(\mu)$ with μ a finite measure,
  - Y is uniformly convex,
  - $Y = C(K)$ for K a compact Haurdorff space.

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- $(L_1[0,1], L_\infty[0,1])$ has the **BPBp** (2011, R. Aron, Y. S. Choi, D. García, M. Maestre)
- (X, A) has the **BPBp** (X Asplund and A uniform algebra) (2013, B. Cascales, A. Guirao, V. Kadets)

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- $(L_1(\mu), L_1(\nu))$ has the **BPBp**
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Pairs of classic Banach spaces having the BPBp

	RANGE SPACES									
DOMAIN SPACES	FD	ℓ_1^n	ℓ_p^n	ℓ_q^n	ℓ_∞^n	c_0	ℓ_1	ℓ_p	ℓ_q	ℓ_∞
FD	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$				$\checkmark$
ℓ_1^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$				$\checkmark$
ℓ_p^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ℓ_q^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ℓ_∞^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$				$\checkmark$
c_0						$\checkmark$				$\checkmark$
ℓ_1	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ℓ_p	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ℓ_q	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ℓ_∞						$\checkmark$				$\checkmark$
$L_1(\mu)$						$\checkmark$	$\checkmark_\sigma$			$\checkmark$
$L_1(\nu)$						$\checkmark$	$\checkmark_\sigma$			$\checkmark$
$L_p(\mu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$L_p(\nu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$L_q(\mu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$L_q(\nu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$L_\infty(\mu)$						$\checkmark$				$\checkmark$
$L_\infty(\nu)$						$\checkmark$				$\checkmark$
$C(K)$			$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$		$\checkmark$		$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	$\checkmark$
$C(S)$			$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$		$\checkmark$		$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	$\checkmark$
$C_0(L_1)$						$\checkmark$				$\checkmark$
$C_0(L_2)$						$\checkmark$				$\checkmark$

Pairs of classic Banach spaces having the BPBp

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		$L_1(\mu)$	$L_1(\nu)$	$L_p(\mu)$	$L_p(\nu)$	$L_q(\mu)$	$L_q(\nu)$
DOMAIN SPACES	FD						
	ℓ_1^n						
	ℓ_p^n	✓	✓	✓	✓	✓	✓
	ℓ_q^n	✓	✓	✓	✓	✓	✓
	ℓ_∞^n						
	c_0			✓	✓	✓	✓
	ℓ_1	✓	✓	✓	✓	✓	✓
	ℓ_p	✓	✓	✓	✓	✓	✓
	ℓ_q	✓	✓	✓	✓	✓	✓
	ℓ_∞						
	$L_1(\mu)$	✓	✓	✓	✓	✓	✓
	$L_1(\nu)$	✓	✓	✓	✓	✓	✓
	$L_p(\nu)$	✓	✓	✓	✓	✓	✓
	$L_p(\mu)$	✓	✓	✓	✓	✓	✓
	$L_q(\mu)$	✓	✓	✓	✓	✓	✓
	$L_q(\nu)$	✓	✓	✓	✓	✓	✓
	$L_\infty(\mu)$	✓ $\mathbb{C}$	✓ $\mathbb{C}$	✓	✓	✓	✓
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	$C(K)$			✓ $\mathbb{R}$	✓ $\mathbb{R}$	✓ $\mathbb{R}$	✓ $\mathbb{R}$
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	$C_0(L_1)$	✓ $\mathbb{C}$	✓ $\mathbb{C}$	✓ $\mathbb{C}$	✓ $\mathbb{C}$	✓ $\mathbb{C}$	✓ $\mathbb{C}$
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DOMAIN SPACES	FD	$\checkmark_\sigma$	$\checkmark_\sigma$				
	ℓ_1^n						
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	ℓ_∞^n						
	c_0	$\checkmark_\sigma$	$\checkmark_\sigma$			$\checkmark$	$\checkmark$
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	ℓ_q	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_∞						
	$L_1(\mu)$	$\checkmark_\bullet$	$\checkmark_\bullet$				
	$L_1(\nu)$	$\checkmark_\bullet$	$\checkmark_\bullet$				
	$L_p(\nu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_p(\mu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_q(\mu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
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	$C_0(L_1)$					$\checkmark_\circ$	$\checkmark_\circ$
	$C_0(L_2)$					$\checkmark_\circ$	$\checkmark_\circ$

The Bishop-Phelps-Bollobás Point Property

Bishop-Phelps-Bollobás point property (BPBpp)

A pair of Banach spaces (X, Y)

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BPBpp $\Rightarrow$ BPBp

Theorem

The Banach space X is uniformly smooth if and only if the pair $(X; \mathbb{K})$ has the BPBpp.

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Examples:

(a) If H is a Hilbert space, then the pair $(H; \mathbb{K})$ has the BPBpp.

(b) The pair $(L_p(\mu); \mathbb{K})$ has the BPBpp for a σ-finite measure μ and $1 < p < \infty$.

Theorem

Let X be a Banach space. Suppose that there is some Banach space Y such that the pair $(X; Y)$ has the BPBpp. Then X is uniformly smooth.

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(a) The pair $(c_0; Y)$ **fails** the BPBpp for all Banach space Y.

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Assume that X is uniformly smooth and that Y has the property β. Then the pair $(X; Y)$ has the BPBpp.

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(a) The pairs $(L_p(\mu); c_0)$ and $(L_p(\mu); \ell_\infty)$ have the BPBpp for a σ-finite measure μ and $1 < p < \infty$.

(b) If H is a Hilbert space, then the pairs $(H; c_0)$ and $(H; \ell_\infty)$ has the BPBpp.

On norm attaining theory and Bishop-Phelps-Bollobás properties

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Centrum pokročilých aplikovaných přírodních věd

(Center for Advanced Applied Science)

February, 2020
Tartu, Estonia

Recap

Definition

$x^* \in X^*$ is **norm attaining** if $\|x^*\| = |x^*(x_0)|$ for some $x_0 \in S_X$.

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Question

When does the set $\mathrm{NA}(X)$ is norm dense in X^*?

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Bishop-Phelps Theorem (1961)

For every Banach space, $\overline{\text{NA}(X)} = X^*$.

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Lindenstrauss

There is X such that $\overline{\text{NA}(X,X)} \neq \mathcal{L}(X,X)$.

Bishop-Phelps-Bollobás theorem (1970)

Let $\varepsilon \in (0,2)$. Given $(x, x^*) \in B_X \times B_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

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Remark

There is no Bishop-Phelps-Bollobás theorem for operators.

Bishop-Phelps-Bollobás property (BPBp)

A pair of Banach spaces (X, Y) is said to have the **BPBp** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

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(X, Y) satisfies the BPBp $\Rightarrow \overline{\mathrm{NA}(X, Y)} = \mathcal{L}(X, Y)$.

The Bishop-Phelps-Bollobás Point Property

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BPBpp $\Rightarrow$ BPBp

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Let H be a real or complex Hilbert space and assume that $u, v \in S_H$. Then, there is a surjective linear isometry R on H such that

$$R(u) = v \quad \text{and} \quad \|R - Id_H\| = \|u - v\|.$$

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(D., S.K. Kim, and H.J. Lee, 2016)

(H, Y) satisfies the BPBpp, for every Banach Y.

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Suppose that there is some Banach space Y such that the pair (X, Y) has the BPBpp. Then, X is uniformly smooth.

Examples:

(a) (H, Y) has the BPBpp, for every Y.

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If X is uniformly smooth and Y has the property β, then (X, Y) has the BPBpp.

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Examples:

(a) $(L_p(\mu), c_0)$ has the BPBpp for σ-finite μ, $1 < p < \infty$.

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Question

$\exists$ uniformly smooth X such that (X, Y) fails the BPBpp for some Y?

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(D., S.K. Kim, and H.J. Lee, 2016)

Taking X with $\dim(X) = 2$, uniformly smooth, not strictly convex, then there is Y such that (X, Y) **fails** the BPBp.

The Bishop-Phelps-Bollobás Operator Property

(S.K. Kim and H.J. Lee, 2014)

A Banach space is uniformly convex if and only given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x \in B_X$ satisfy

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

there is $x_0 \in S_X$ such that

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Questions

How about operators?

Bishop-Phelps-Bollobás operator property (BPBop)

A pair of Banach spaces (X, Y) is said to have the **BPBop** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

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Questions

What happens with the pair $(\ell_2^2, \ell_\infty^2)$?

All the following pairs **fail** to have the BPBop (D., 2017):

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(9) (X, X) for $\dim(X) = 2$.

(D., V. Kadets, S.K. Kim, H.J. Lee, M. Martín, 2018)

$\dim(X), \dim(Y) > 1 \Rightarrow (X, Y)$ fails the BPBop.

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$\dim(X), \dim(Y) > 1 \Rightarrow (X, Y)$ fails the BPBop.

- The proof reduces to the case $\dim(X) = \dim(Y) = 2$.
- We construct a $T \in S_{\mathcal{L}(X,Y)}$ such that $T(B_X)$ has an special "position" in Y.

LOCAL BISHOP-PHELPS-BOLLOBÁS PROPERTIES

Local Bishop-Phelps-Bollobás Operator Property

(2017, D.) Local BPBop

A pair of Banach spaces (X, Y) is said to have the **local BPBop** if for every $\varepsilon \in (0,1)$ and $T \in S_{\mathcal{L}(X,Y)}$, there exists $\eta(\varepsilon, T) > 0$ such that if $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon, T),$$

there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

- $\dim(X) < \infty \Rightarrow (X, Y)$ has the local BPBop $\forall Y$. (2017, D.)

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- (ℓ_p, ℓ_q) has the local BPBop whenever $1 \leq q < p < \infty$.
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- If X is a reflexive space with the Kadec-Klee property, then (X, Y) has the local BPBop for compact operators $\forall Y$.
  (2018, D. Sain)

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- (ℓ_p, ℓ_q) has the local BPBop whenever $1 \leq q < p < \infty$.
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- If X is a reflexive space with the Kadec-Klee property, then (X, Y) has the local BPBop for compact operators $\forall Y$.
  (2018, D. Sain)
- It was characterized in the setting with strictly convex domain spaces and compact operators.
  (2018, J. Talponen)

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$$\|S(x)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

We say that the norm of X is strongly subdifferentiable (SSD, for short) at a point $u \in S_X$ if the one-sided limit

$$\lim_{t \to 0^+} \frac{1}{t}(\|u + tx\| - 1)$$

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- Consider ϕ_n on B_X defined by

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- Then, X is SSD iff $\{\phi_n\}$ converges uniformly on B_X.

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- The norm of ℓ_1 is only SSD at points in S_{ℓ_1} which are sequences with finitely many nonzero terms.
  (J.R. Giles, D A. Gregory, B. Sims, 1978)

SSD

- The norm of any finite-dimensional space is SSD.
  (Dini's theorem)

- The sup-norm on c_0 is SSD at every point of c_0.
  (C. Franchetti, 1986)

- The set of all SSD points of ℓ_∞ is not a G_δ in ℓ_∞.
  (G. Godefroy, V. Montesinos, V. Zizler, 1995)

- The norm of ℓ_1 is only SSD at points in S_{ℓ_1} which are sequences with finitely many nonzero terms.
  (J.R. Giles, D A. Gregory, B. Sims, 1978)

- A Banach space with an SSD norm is Asplund.
  (C. Franchetti and R. Payá, 1993)
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Theorem (C. Franchetti and R. Payá, 1993)

The pair $(X, \mathbb{K})$ has the local BPBpp iff X is SSD.

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are non-reflexive dual spaces that satisfy the w^*-Kadec-Klee property.

Property $\star$ for operators

Examples of linear operators
(D., S.K. Kim, H.J. Lee, M. Mazzitelli, 2018)

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Strongly Subdifferentiability in projective tensor products

SSD in projective tensor products

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SSD in projective tensor products

As a consequence... (D., S.K. Kim, H.J. Lee, M. Mazzitelli, 2020)

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SSD in projective tensor products

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(c). If $p^{-1} + q^{-1} \geq 1$ or one of them is 1 or ∞, then $\ell_p \hat{\otimes}_\pi \ell_q$ is **not** SSD.

Relations between the properties

$$
\begin{tikzcd}
\textbf{BPBpp} \arrow[r] \arrow[dr] & \text{local } \textbf{BPBpp} \arrow[r] & \mathbf{L}_p & \\
& \textbf{BPBp} \arrow[ur] \arrow[dr] & & \overline{\mathrm{NA}} = \mathcal{L} \\
\textbf{BPBop} \arrow[r] \arrow[ur] & \text{local } \textbf{BPBop} \arrow[r] & \mathbf{L}_o \arrow[ur] &
\end{tikzcd}
$$

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Q2. $\exists$ more Banach spaces X and Y such that $X \hat{\otimes}_\pi Y$ is SSD?

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Q3. $(L_p(\mu), Y)$ satisfies the BPBpp, $1 < p < 2, \forall Y$?

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Q4. (ℓ_p^2, ℓ_q) has the local BPBpp for $1 < p, q < \infty$?

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Is it also true for complex-valued $C(K)$ spaces?

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Q9. Does property A implies the Radon-Nikodým property?

Questions

Q10. $\mathbb{R}^2$ has property B? That is, $\overline{\mathrm{NA}(X, \mathbb{R}^2)} = \mathcal{L}(X, \mathbb{R}^2), \forall X$?

Thank you
for your attention