Super Bishop-Phelps-Bollobás property

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Every element in X^* can be approximated by a norm attaining linear functional. In other words, $\overline{NA(X)} = X^*$.

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Question (Bishop-Phelps)

Is it true for operators?

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We say that a bounded linear operator $T \in \mathcal{L}(X,Y)$ **attains its norm** if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\|$. The set of all norm attaining operators is denoted by $NA(X,Y)$.

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Lindenstrauss' counterexample, 1963

There exists a Banach space X such that

$$\overline{NA(X, X)} \neq \mathcal{L}(X, X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators.

Definitions & Some Results

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Bishop-Phelps-Bollobás Theorem, 1970 (Martín's version, 2014)

Let X be a Banach space and $\varepsilon \in (0, 2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$\operatorname{Re} x^*(x) > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = y^*(y) = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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Observation

It is **not** expected that there exists a Bishop-Phelps-Bollobás Theorem version for bounded linear operators.

Definitions & Some Results

In 2008, Acosta, Aron, García and Maestre introduced the following property:

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Definition - Bishop-Phelps-Bollobás property (BPBp)

A pair of Banach spaces (X, Y) is said to have the **BPBp** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

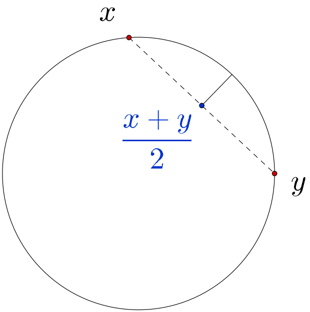
there exist $S \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

Super BPBp

A Banach space X is **uniformly convex** if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$x, y \in S_X \text{ and } \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon).$$



Super BPBp

In 2014, Kim and Lee proved that

Kim-Lee Theorem

A Banach space X is **uniformly convex** if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x \in B_X$ satisfy

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

there is $x_0 \in S_X$ such that

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Question

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Definition of the sBPBp

We say that the pair of Banach spaces (X, Y) has the **super BPBp** if for every $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exists $x_0 \in S_X$ such that

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$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

By the Kim-Lee Theorem, the pair $(X, \mathbb{K})$ has the sBPBp if and only if X is a uniformly convex Banach space.

Super BPBp

D., García, Maestre and Martín

Suppose that X is a Banach space. If the pair (X, Y_0) has the sBPBp **for some** Banach space Y_0, then the pair $(X, \mathbb{K})$ has the sBPBp.

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By the Kim-Lee Theorem, we have the following result:

Corollary

Suppose that X is a Banach space. If the pair (X, Y_0) has the sBPBp **for some** Banach space Y_0, then X is uniformly convex.

Super BPBp

Question

Suppose, **for example**, that X is a **uniformly convex** (or a **Hilbert**) space and Y is **any** (or a **uniformly convex** or a **Hilbert**) space.

Given $\varepsilon \in (0,1)$, is it possible to find a positive real number $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x - x_0\| < \varepsilon?$$

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Counterexample

Consider $X = \ell_2^2(\mathbb{K})$ and $Y = \ell_\infty^2(\mathbb{K})$.

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$$T(x, y) := \left(\left(1 - \frac{1}{2}\eta(\varepsilon) \right) x, y \right).$$

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So:

- $\|T\| = 1$,
- $\|T(e_1)\|_\infty > 1 - \eta(\varepsilon)$,
- every $z \in S_X$ such that $\|T(z)\|_\infty = 1$ assumes the form $z = \lambda e_2$ for some $|\lambda| = 1$.

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But, in this case, we have $\|e_1 - z\|_2 = \sqrt{2}$.

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What if we add on the hypothesis of the problem that both X and Y are uniformly convex Banach spaces?

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If we take $X = Y = \ell_2^2(\mathbb{R})$ and $T : X \to X$ defined by

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then we obtain the **same contradiction** again:

- $\|T\| = 1$,
- $\|T(e_1)\|_2 > 1 - \eta(\varepsilon)$,
- $z \in S_X$ and $\|T(z)\|_2 = 1 \implies z = \lambda e_2$ with $|\lambda| = 1$.
- $\|z - e_1\|_2 = \sqrt{2}$.

Super BPBp

More in general, we have the following positive result.

D., García, Maestre, Martín, 2015

Let ℓ_p^2 and ℓ_q^2 be the space $\mathbb{K}^2$ endowed with the norms $\| \, . \, \|_p$ and $\| \, . \, \|_q$, respectively, with $1 < p \leq q < \infty$ (or $p < q = \infty$).

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(i) $\|T_\beta\| = 1$,

(ii) $\|T_\beta(e_1)\|_q = \beta$ and

(iii) for every $z \in S_{\ell_p^2}$ such that $\|T_\beta(z)\|_q = 1$, we have $\|z - e_1\|_p = 2^{\frac{1}{p}}$.

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- The **Kim-Lee Theorem** says that the pair $(X, \mathbb{K})$ has the sBPBp if and only if X is uniformly convex.

- the pairs $(\ell_2^2, \ell_\infty^2)$, (ℓ_2^2, ℓ_2^2) and (ℓ_p^2, ℓ_q^2) with $1 < p \leq q < \infty$ fail this property.

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So $\|e_1 - z\|_2 = \sqrt{2}$

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More in general, we have the following result:

D., García, Maestre, Martín, 2015

Let ℓ_2^2 and ℓ_q^2 be the space $\mathbb{R}^2$ endowed with the norms $\| \, . \, \|_2$ and $\| \, . \, \|_q$, respectively, with $1 \leq q < 2$.

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(ii) for every $z \in S_{\ell_2^2}$ such that $\|T_\beta(z)\|_q = 1$, we have $\|z - e_1\|_2 = \sqrt{2}$.

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(ii) for every $z \in S_{\ell_2^2}$ such that $\|T_\beta(z)\|_q = 1$, we have $\|z - e_1\|_2 = \sqrt{2}$.

Corollary

The pair $(\ell_2^2(\mathbb{R}), \ell_q^2(\mathbb{R}))$ fails the sBPBp for $1 \leq q \leq \infty$.

Super BPBp

If we use similar ideas of the counter examples above, we can also prove that

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D., García, Maestre, Martín, 2015

The pair (X, ℓ_∞^2) fails the sBPBp for all Banach spaces X.

Super BPBp

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Let X be a Banach space. If there exists a Banach space Y_0 such that the pair (X, Y_0) has the sBPBp, then the pair X is uniformly convex.

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Another result is the following: by the result

D., García, Maestre, Martín, 2015

Let X be a Banach space. If there exists a Banach space Y_0 such that the pair (X, Y_0) has the sBPBp, then the pair X is uniformly convex.

and since ℓ_1^2 is not a uniformly convex space, then the pair (ℓ_1^2, Y) fails the sBPBp for all Banach spaces Y.

Super BPBp

To finish this part, by using Auerbach basis, we may prove the following result:

D., García, Maestre, Martín, 2015

The pair (Y, Y) fails the sBPBp for all 2-dimensional normed space Y.

Super BPBp

Definition of the sBPBp

We say that the pair of Banach spaces (X, Y) has the **super BPBp** if for every $\varepsilon \in (0,1)$, there exists $\boxed{\eta(\varepsilon) > 0}$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \boxed{\eta(\varepsilon)},$$

there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

Weak sBPBp

Definition of the weak sBPBp

We say that the pair of Banach spaces (X, Y) has the **weak super BPBp** if for every $\varepsilon \in (0,1)$, there exists $\boxed{\eta(\varepsilon, T) > 0}$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \boxed{\eta(\varepsilon, T)},$$

there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

Weak sBPBp

D., García, Maestre, Martín, 2015

Suppose that X and Y are Banach spaces with $\dim X < \infty$. Fix $T_0 \in S_{\mathcal{L}(X,Y)}$ and let $\varepsilon \in (0,1)$. Then there exists $\eta(\varepsilon, T_0) > 0$ such that whenever $x \in S_X$ satisfies

$$\|T_0(x)\| > 1 - \eta(\varepsilon, T_0),$$

there exists $x_0 \in S_X$ such that

$$\|T_0(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

Weak sBPBp

D., García, Maestre, Martín, 2015

Let X be a **uniformly convex** Banach space and Y be **any** Banach space. Fix $T_0 : X \to Y$ be a **compact operator** with $\|T_0\| = 1$ and let $\varepsilon > 0$. Then there exists $\eta(\varepsilon, T_0) > 0$ such that whenever $x \in S_X$ satisfies

$$\|T_0(x)\| > 1 - \eta(\varepsilon, T_0)$$

there exists $x_0 \in S_X$ such that

$$\|T_0(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

Thank you very much
for your attention.