The Bishop-Phelps-Bollobás point property

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Cartagena, Spain, June 10th, 2016

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Motivation

The Bishop-Phelps Theorem (1961)

The set of all norm attaining functionals $\mathrm{NA}(X)$ is dense in X^*.

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Question: Is this true for bounded linear operators?

Lindenstrauss' counterexample, 1963

There exists a Banach space X such that

$$\overline{NA(X,X)} \neq \mathcal{L}(X,X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators.

Motivation

The Bishop-Phelps-Bollobás theorem (1970) [Martín's version]

Let X be a Banach space. Let $0 < \varepsilon < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy

$$\operatorname{Re} x^*(x) > 1 - \frac{\varepsilon^2}{2}.$$

Then, there are $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(y)| = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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Remark: The Bishop-Phelps-Bollobás theorem implies the Bishop-Phelps theorem.

Motivation

In 2008, Acosta, Aron, García and Maestre introduced

The Bishop-Phelps-Bollobás property

A pair of Banach spaces (X, Y) is said to have the **BPBp** if given $\varepsilon > 0$, then there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_1 \in S_X$ such that

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Motivation

All the following pairs have the BPBp:

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All the following pairs have the BPBp:

- $(\mathbb{K}^n, \mathbb{K}^m)$ for $n, m \in \mathbb{N}$,
- $(\ell_1, C(K))$ for a compact Hausdorff topological space K,
- $(L_p(\mu), c_0)$ for every $1 < p < \infty$,
- (H_1, H_2) whenever H_1 and H_2 are Hilbert spaces.
- $(C_0(L), L_p(\mu))$ for every Hausdorff locally compact space L and $1 \leq p < \infty$.

The Bishop-Phelps-Bollobás point property

Kim-Lee Theorem (2014)

A Banach space X is **uniformly convex** if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x \in B_X$ satisfy

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

there is $x_0 \in S_X$ such that

$$|x^*(x_0)| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

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there is $x_0 \in S_X$ such that

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Question Is it true for operators?

The Bishop-Phelps-Bollobás point property

The pair (ℓ_2^2, ℓ_2^2) **does not** satisfy this property. (D., 2016)

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Question What if we do not change the initial point?

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Question What if we do not change the initial point?

The Bishop-Phelps-Bollobás point property

A pair (X, Y) is said to have the **BPBpp** if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there exists $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ such that

$$\|S(x_0)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

The Bishop-Phelps-Bollobás point property

Theorem

The Banach space X is uniformly smooth if and only if the pair $(X, \mathbb{K})$ has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

The Banach space X is uniformly smooth if and only if the pair $(X, \mathbb{K})$ has the BPBpp.

Examples:

(a) If H is a Hilbert space, then the pair $(H, \mathbb{K})$ has the BPBpp.

(b) The pair $(L_p(\mu), \mathbb{K})$ has the BPBpp for a σ-finite measure μ and $1 < p < \infty$.

The Bishop-Phelps-Bollobás point property

Theorem

Let X be a Banach space. Suppose that there is some Banach space Y such that the pair (X, Y) has the BPBpp. Then X is uniformly smooth.

The Bishop-Phelps-Bollobás point property

Theorem

Let X be a Banach space. Suppose that there is some Banach space Y such that the pair (X, Y) has the BPBpp. Then X is uniformly smooth.

Examples:

(a) The pair (c_0, Y) fails the BPBpp for all Banach space Y.

(b) The pair (ℓ_1, Y) fails the BPBpp for all Banach space Y.

The Bishop-Phelps-Bollobás point property

Theorem

Assume that X is uniformly smooth and that Y has the property β. Then the pair (X, Y) has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

Assume that X is uniformly smooth and that Y has the property β. Then the pair (X, Y) has the BPBpp.

Examples:

(a) The pairs $(L_p(\mu), c_0)$ and $(L_p(\mu), \ell_\infty)$ have the BPBpp for a σ-finite measure μ and $1 < p < \infty$.

(b) If H is a Hilbert space, then the pairs (H, c_0) and (H, ℓ_∞) has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

Let H be a Hilbert space and let Y be any Banach space. Then the pair (H, Y) has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

Let H be a Hilbert space and let Y be any Banach space. Then the pair (H, Y) has the BPBpp.

Open problem: The pair (X, H) has the BPBpp when X is any Banach space and H is a Hilbert space?

The Bishop-Phelps-Bollobás point property

Theorem

Let X be a uniformly smooth Banach space and A be a uniform algebra. The pair (X, A) has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

Let X be a uniformly smooth Banach space and A be a uniform algebra. The pair (X, A) has the BPBpp.

Examples:

(a) The pair $(L_p(\mu), C(K))$ has the BPBpp.

(b) The pair $(H, C(K))$ has the BPBpp.

The Bishop-Phelps-Bollobás point property

Counterexample with X uniformly smooth

Kim and Lee proved that a 2-dimensional real Banach space X is uniformly convex if and only if the pair (X, Y) has the BPBp for all Banach spaces Y.

So, if X_0 is a 2-dimensional real Banach space which is uniformly smooth but not strictly convex, there exists a Banach space Y_0 such that the pair (X_0, Y_0) do not have the BPBp and thus it fails the BPBpp.

The BPBpp for bilinear mappings

The BPBp for bilinear mappings

We say that a pair of Banach spaces $(X \times Y, Z)$ has the **BPBpp for bilinear mappings** if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $B \in B(X \times Y, Z)$ with $\|B\| = 1$ and $(x_0, y_0) \in S_X \times S_Y$ satisfy

$$\|B(x_0, y_0)\| > 1 - \eta(\varepsilon),$$

there exists $A \in B(X \times Y, Z)$ with $\|A\| = 1$ such that

$$\|A(x_0, y_0)\| = 1 \quad \text{and} \quad \|A - B\| < \varepsilon.$$

The BPBpp for bilinear mappings

The BPBpp for bilinear mappings

(a) Let X be uniformly smooth and H a Hilbert space. The pair $(X \times H, \mathbb{K})$ has the BPBpp for bilinear forms if and only if the pair (X, H^*) has the BPBpp (for operators).

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(a) Let X be uniformly smooth and H a Hilbert space. The pair $(X \times H, \mathbb{K})$ has the BPBpp for bilinear forms if and only if the pair (X, H^*) has the BPBpp (for operators).

(b) The pair $(H_1 \times H_2, \mathbb{K})$ has the BPBpp for bilinear forms.

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(a) Let X be uniformly smooth and H a Hilbert space. The pair $(X \times H, \mathbb{K})$ has the BPBpp for bilinear forms if and only if the pair (X, H^*) has the BPBpp (for operators).

(b) The pair $(H_1 \times H_2, \mathbb{K})$ has the BPBpp for bilinear forms.

(c) Let Z be a Banach space with property β. Then the pair $(H_1 \times H_2, Z)$ has the BPBpp for bilinear mappings.

Thank you for your attention!