Smooth norms in dense subspaces of Banach spaces

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(ii) *If the dual norm is Fréchet differentiable, then* X *is reflexive.*

(iii) *If the dual norm on* X^* *is LUR, then* $\|\cdot\|$ *is Fréchet differentiable.*

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Theorem (M. Fabian, 1987) *If a Banach space X admits a C^1-smooth bump, then it is Asplund.*

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Question *Does every Asplund Banach space admit a C^1-smooth bump function?*

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- (V.Z. Meshkov, 1978) *If* X *and* X^* *admit a* C^2*-smooth norm, then* X *is isomorphic to a Hilbert space.*
- (M. Fabian, J.H.M. Whitfield, and V. Zizler, 1983) *If* X *admits a* C^2*-smooth renorming then* X *either contains a copy of* c_0 *or it is superreflexive.*
- (R. Deville, 1989) *The existence of a* C^∞*-smooth norm on a Banach space* X *that contain no copy of* c_0 *implies that* X *is of cotype* $2k$*, for some* k*, and it contains* ℓ_{2k}*.*

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- (Hájek, 1995) If X is separable, yes!

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Let Γ *be an uncountable set and* $\mathcal{F}$ *be a normed space of all finitely supported vectors in* $\ell_1(\Gamma)$ *endowed with the* ℓ_1*-norm. Does* $\mathcal{F}$ *admit a Fréchet smooth norm?*

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Q3. What can one say about the whole space X if there exists a dense subspace Y which admits a C^k-smooth norm?

The results

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- A function $f : U \to Y$ is **analytic** if $f(x) = \sum_n P_n(x-a)$.
- Given a normed space $(X, \|\cdot\|)$ and ε, we say that a new norm $|||\cdot|||$ **ε-approximates** $\|\cdot\|$ if $|||\cdot||| \leq \|\cdot\| \leq (1+\varepsilon)|||\cdot|||$.

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Then, the Minkowski functional μ on B is an equivalent C^k-smooth norm on X.

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(D., Hájek, Russo)

The space ℓ_∞^F admits an analytic norm that approximates $\|\cdot\|_\infty$.

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Consequence 1.1

Let X be a separable Banach space. Then there is a dense subspace Y of X which admits an analytic norm that approximates the original norm of Y.

The results

Consequence 2

The normed space $(F, \|\cdot\|_1)$ of all finitely supported vectors in $\ell_1(\mathfrak{c})$, where $\mathfrak{c}$ denotes a set of cardinality continuum, admits an analytic norm that approximates $\|\cdot\|_1$.

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The normed space $(F, \|\cdot\|_1)$ of all finitely supported vectors in $\ell_1(\mathfrak{c})$, where $\mathfrak{c}$ denotes a set of cardinality continuum, admits an analytic norm that approximates $\|\cdot\|_1$.

Consequence 2.2

Let S be a set with $|S| = \Gamma \leq \mathfrak{c}$. Then, every dense subspace of $\ell_1(S)$ contains a further dense subspace which admits an analytic norm.

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No dense subspace of $c_0(\omega_1)$ admits an analytic norm.

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A set $\{e_\gamma\}_{\gamma \in \Gamma}$ in a Banach space X is called an **unconditional Schauder basis** of X if for every $x \in X$ there is a unique family of real numbers $\{a_\gamma\}_{\gamma \in \Gamma}$ such that $x = \sum_{\gamma \in \Gamma} a_\gamma x_\gamma$ in the following sense: for every $\varepsilon > 0$, there is a finite subset $F \subset \Gamma$ such that

$$\left\| x - \sum_{\gamma \in G} a_\gamma e_\gamma \right\| < \varepsilon,$$

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$$\left\| x - \sum_{\gamma \in G} a_\gamma e_\gamma \right\| < \varepsilon,$$

whenever $F \subset G$. If $\{e_\gamma\}_{\gamma \in \Gamma}$ is an unconditional basis of X and A is a subset of Γ, then there is a naturally defined bounded linear projection P_A from X onto $\overline{\text{span}}\{e_\gamma\}_{\gamma \in A}$ defined by $P_A(x) = \sum_{\gamma \in A} \langle e_\gamma^*, x \rangle e_\gamma$.

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The results

(D., Hájek, Russo)

Let X be a Banach space with a suppression 1-unconditional Schauder basis $\{e_\gamma\}_{\gamma \in \Gamma}$ and set $Y = \text{span}\{e_\gamma\}_{\gamma \in \Gamma}$. Then, Y is a dense subspace of X which admits a C^∞-smooth norm that approximates the original one.

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Corollary 1

The linear span of the canonical basis of $\ell_p(\Gamma)$ for $1 \leq p < \infty$ admits a C^∞-smooth norm.

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The linear span of the canonical basis of $\ell_p(\Gamma)$ for $1 \leq p < \infty$ admits a C^∞-smooth norm.

Corollary 2

Every dense subspace of $\ell_1(\Gamma)$ contains a further dense subspace which admits a C^∞-smooth norm.

Some questions and further study

Q1. *Is there a Banach space such that no dense subspace admit a C^k-smooth norm?*

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The space $\ell_\infty^{c,F}(S)$ of finitely-valued sequences in $\ell_\infty^c(S)$ does not admit a Gâteaux differentiable norm, whenever S is an uncountable.

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Q3. *If a Banach space X is such that every dense subspace contains a further dense subspace with a C^k-smooth norm, what can we say about the whole space X?*

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Q4. *In our results, we got a smooth norm on* **one** *specific dense subspace. How about other dense subspaces?*

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Q5. *How different can two dense subspaces of a Banach space be?*

Thank you
for your attention