On Banach spaces whose group of isometries acts micro-transitively

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Notation

- X, Y are real or complex Banach spaces
- $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$
- B_X is the closed unit ball of X
- S_X is the unit sphere of X
- $\mathcal{L}(X, Y)$ continuous linear operators from X into Y
  - If $X = Y$, then $\mathcal{L}(X, X) = \mathcal{L}(X)$
- $\mathcal{G}(X)$ all surjective linear isometries from X to X

Motivation

Banach-Mazur rotation problem

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Is every transitive separable Banach space isometrically isomorphic to a Hilbert space?

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(i) $ex = x, \forall x \in T$,

(ii) $g(hx) = (gh)x, \forall g, h \in G, \forall x \in T$.

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(i) If the action is micro-transitive, then the orbits of the elements are open. (F.D. Ancel, 1987)

(ii) So, they produce a partition of the space T as a disjoint union of open sets.

(iii) Therefore, if T is connected, micro-transitivity $\Rightarrow$ transitivity.

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Hilbert spaces have the above property with $\beta(\varepsilon) = \varepsilon$.
(see, for example, M. Acosta, M. Mastyło, and M. Soleimani-M., 2018)

Notation

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Definition (Uniform micro-semitransitivity)

We say that the norm of X is **uniformly micro-semitransitive** if there is a function $\beta : (0, 2) \longrightarrow \mathbb{R}^+$ such that whenever $x, y \in S_X$ satisfies

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Indeed, we take $x_1, \ldots, x_n \in S_X$ such that

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(c) For every Banach space X, given $x, y \in S_X$, there is $T \in \mathcal{L}(X)$ with $\|T\| = 1$ such that $Tx = y$.

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(c) For every Banach space X, given $x, y \in S_X$, there is $T \in \mathcal{L}(X)$ with $\|T\| = 1$ such that $Tx = y$.

(d) If X is uniformly micro-semitransitive and Y is a 1-complemented subspace of X, then Y is uniformly micro- semitransitive.

The BPBp

The Bishop-Phelps-Bollobás property

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Bishop-Phelps-Bollobás property
(M. Acosta, R. Aron, D. García, and M. Maestre, 2008)

A pair (X, Y) of Banach spaces has the **BPBp** if for every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x \in S_X$ such that

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(a) (Bollobás, 1963) $(X, \mathbb{K})$ has the **BPBp**, for all Banach X.

(b) (S.K. Kim and H.J. Lee, 2014) If X is uniformly convex, then (X, Y) has the **BPBp** for all Banach Y.

The BPBp

Bishop-Phelps-Bollobás point property (D., S.K. Kim, and H.J. Lee (2016))

We say that a pair (X, Y) of Banach spaces has the **BPBpp** if given $\varepsilon > 0$, there exists $\tilde{\eta}(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy

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(a) The pair $(X, \mathbb{K})$ has the **BPBpp** if and only if X is uniformly smooth.

(b) If $\exists Y$ such that (X, Y) has the **BPBpp**, then X is uniformly smooth.

The BPBp

(D., V. Kadets, S.K. Kim, H.J. Lee, and M. Martín (2018))

(a) If (X, Y) has the **BPBpp** $\forall Y$, then X is uniformly convex.

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(a) If (X, Y) has the **BPBpp** $\forall Y$, then X is uniformly convex. Moreover,

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for suitable $2 \leq q < \infty$ and $C > 0$.

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(D., V. Kadets, S.K. Kim, H.J. Lee, and M. Martín (2018))

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(Transitive + superflexive $\Rightarrow$ uniformly convex (Finet, 1986))

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If, moreover, X is isomorphic to a Hilbert space, then the above conditions are indeed equivalent to

(e) $\exists\ C > 0$ such that $\delta_X(\varepsilon) \geq C\,\varepsilon^2$ $(0 < \varepsilon < 2)$, where the constant C depends only on the modulus of convexity of X, on the function $\beta(\cdot)$ of the definition of uniform micro-transitivity, and on the Banach-Mazur distance from X to the Hilbert space.

Corollary 5

If X is micro-transitive, then

(a) For every Banach space Y, the pair (X, Y) has the **BPBpp**,

(b) there exists $2 \leq q < \infty$ and $C > 0$ so that $\delta_X(\varepsilon) \geq C\,\varepsilon^q$ for $0 < \varepsilon < 2$.

If, moreover, X is isomorphic to a Hilbert space, then

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Remark: For $1 \leq p < \infty$, there are (non-separable) L_p-spaces whose standard norms are transitive but **they are not** micro-transitive unless $p = 2$.

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Indeed, for $p > 2$, there is **no** isomorphism $T \in \mathcal{L}(\ell_p^2)$ with $\|T\| = 1$ such that

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Since ℓ_p^2 is always 1-complemented in $L_p(\mu)$, we are done.

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Every space that is of type 2 and cotype 2 is isomorphic to a Hilbert space (S. Kwapień, 1972)

Thank you
for your attention