# On the Bishop-Phelps-Bollobás point property

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**The 13th ILJU School of Mathematics**
**Banach Spaces and Related Topics**

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January 30th, 2018, Daejeon, South Korea

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# Notation

$X, Y$ and $Z$ are real or complex Banach spaces.

- $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$,
- $B_X$ is the closed unit ball of $X$,
- $S_X$ is the unit sphere of $X$,
- $\mathcal{L}(X, Y)$ continuous linear operators from $X$ into $Y$,
- $K(X, Y)$ compact linear operators from $X$ into $Y$,
- $X^* = \mathcal{L}(X; \mathbb{K})$ topological dual of $X$.

# Motivation & History background

**Definition**

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NA($X$) is the set of all **norm attaining functionals**.

**James theorem (1957)**

A Banach space $X$ is **reflexive** if and only if every bounded linear functional is norm attaining.

# Motivation & History background

**Bishop-Phelps theorem (1961)**

Every element in $X^*$ can be approximated by a norm attaining linear functional. In other words, $\overline{\mathrm{NA}(X)} = X^*$.

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**Question (Bishop-Phelps)**

Is it true for bounded linear operators?

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**(1963, Lindenstrauss) Counterexample**

There exists a Banach space X such that

$$\overline{\text{NA}(X, X)} \neq \mathcal{L}(X, X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators.

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**1970, Bollobás, Bishop-Phelps-Bollobás theorem**
**(2014, M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido)**

Let $\varepsilon \in (0, 2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = |y^*(y)| = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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**Observation 1**

Bishop-Phelps-Bollobás theorem $\Rightarrow$ Bishop-Phelps theorem.

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Bishop-Phelps-Bollobás theorem $\Rightarrow$ Bishop-Phelps theorem.

**Observation 2**

It is **not** expected that there exists a Bishop-Phelps-Bollobás theorem version for bounded linear operators in general.

# Motivation & History background

(2008, M. Acosta, R. Aron, D. García, M. Maestre)

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**Bishop-Phelps-Bollobás property (BPBp)**

A pair of Banach spaces $(X, Y)$ is said to have the **BPBp** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exist $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

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- $Y = L_1(\mu)$ with $\mu$ a finite measure.
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- $Y = C(K)$ for $K$ a compact Haurdorff space.

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  (2014, S. K. Kim, H. J. Lee)

- $(C(K), L_1(\mu))$ has the BPBp.
  (2016, M. Acosta)

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It is clear that BPBpp $\Rightarrow$ BPBp.

# First results about the BPBpp

(2016, D., S. K. Kim and H. J. Lee)

- $(X, \mathbb{K})$ has the BPBpp if and only if $X$ is uniformly smooth.

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- $(H, Y)$ has the BPBpp for all Hilbert spaces $H$ and any $Y$.
- $(X, Y)$ has the BPBpp for $X$ uniformly smooth and $Y$ property $\beta$.
- there are uniformly smooth Banach spaces $X$ such that the pair $(X, Y)$ **fails** the BPBpp for some $Y$.

# Recent results

# Recent results about the BPBpp

## Stability results

**Proposition**

Let $X_1$ be a one-complemented subspace of $X$. If $(X, Y)$ has the BPBpp, then $(X_1, Y)$ has the BPBpp.

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**Proposition**

Let $X_1$ be a one-complemented subspace of $X$. If $(X, Y)$ has the BPBpp, then $(X_1, Y)$ has the BPBpp.

**Questions**

**(a)** Is this true for the BPBp? (2017, D., PhD thesis)

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**(a)** Is this true for the BPBp? (2017, D., PhD thesis)

**(b)** Is this true for norm attaining operators?

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## Stability results

**Proposition ((2017, D., PhD thesis) adapted)**

If $Y = Y_1 \oplus_a Y_2$ and $(X, Y)$ has the BPBpp, then $(X, Y_j)$ has the BPBpp.

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**Proposition ((2017, D., PhD thesis) adapted)**

If $Y = Y_1 \oplus_a Y_2$ and $(X, Y)$ has the BPBpp, then $(X, Y_j)$ has the BPBpp.

**Proposition ((2015, Aron, Choi, Kim, Lee, Martín) adapted)**

If $(X, C(K, Y))$ has the BPBpp, then $(X, Y)$ has the BPBpp.

# Recent results about the BPBpp

## Universal properties

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**Definition (2014, Aron, Choi, Kim, Lee, Martín)**

(a) $X$ is **universal BPBpp domain space** if $(X, Y)$ has the BPBpp for all $Y$.

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**Definition (2014, Aron, Choi, Kim, Lee, Martín)**

(a) $X$ is **universal BPBpp domain space** if $(X, Y)$ has the BPBpp for all $Y$.

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**Examples (2016, D., S. K. Kim, H. J. Lee)**

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**Examples (2016, D., S. K. Kim, H. J. Lee)**

- Hilbert spaces are universal BPBpp domain spaces.
- Uniform algebras and Banach spaces with property $\beta$ are universal BPBpp range spaces.

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## Universal properties

**Question**

We know that Hilbert spaces are universal BPBpp domain spaces.

# Recent results about the BPBpp

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We know that Hilbert spaces are universal BPBpp domain spaces.

Is it possible to extend the result for $L_p$-spaces with $1 < p < \infty$?

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## Universal properties

**Theorem**

If $X$ is universal BPBpp domain space, then $X$ is uniformly convex.

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**Theorem**

If $X$ is universal BPBpp domain space and $X$ is isomorphic to a Hilbert space, then $\delta_X(\varepsilon) \geq C\varepsilon^2$.

# Recent results about the BPBpp

## Universal properties

**Corollary**

$L_p(\mu)$ is **not** a BPBpp domain space for $p > 2$.

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$L_p(\mu)$ is **not** a BPBpp domain space for $p > 2$.

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Is $L_p(\mu)$ a BPBpp domain space for $1 < p < 2$?

# Recent results about the BPBpp

## Universal properties

$ACK_\rho$-structure (2017, Cascales, Guirao, Kadets, Soloviova)

**Theorem**

If $Y$ has $ACK_\rho$-structure, then $Y$ is universal BPBpp range space.

# Recent results about the BPBpp

## Universal properties

$ACK_\rho$-structure (2017, Cascales, Guirao, Kadets, Soloviova)

**Theorem**

If $Y$ has $ACK_\rho$-structure, then $Y$ is universal BPBpp range space.

- $C(K)$ and $C_0(L)$ and, more in general, uniform algebras.
- Banach spaces with property $\beta$.
- finite $\ell_\infty$-sums of Banach spaces with $ACK_\rho$-structure.
- $c_0(Y)$, $\ell_\infty(Y)$ when $Y$ has $ACK_\rho$-structure.
- $C(K, Y)$ when $Y$ has $ACK_\rho$-structure.

# Recent results about the BPBpp

## Universal properties

**Counterexample**

For $p \geq 2$, there is a Banach space $X_p$ which is **uniformly convex and uniformly smooth** such that $(X_p, \ell_p^2)$ **fails** the BPBpp.

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### Questions

**(1)** If $Y$ is universal BPBp range space, then $Y$ is universal BPBpp range for uniformly smooth $X$?

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**Questions**

**(1)** If $Y$ is universal BPBp range space, then $Y$ is universal BPBpp range for uniformly smooth $X$?

**(2)** It is not known whether all finite dimensional spaces are universal **BPBp** range spaces or even if they have Lindenstrauss property B.

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## The BPBpp for compact operators

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- $(X, \ell_p(Y)) \Rightarrow (X, L_p(\mu, Y))$ for $1 \leq p < \infty$.

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- $(X, \ell_p(Y)) \Rightarrow (X, L_p(\mu, Y))$ for $1 \leq p < \infty$.
- $(X, Y) \Rightarrow (X, L_\infty(\mu, Y))$
- $(X, Y) \Rightarrow (X, C(K, Y))$.

# The dual property

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Recall that $(X, Y)$ has the **BPBpp** if $\forall\ \varepsilon \in (0,1)$, $\exists\ \eta(\varepsilon) > 0$:

$$T \in S_{\mathcal{L}(X,Y)}, x \in S_X \text{ with } \|T(x)\| > 1 - \eta(\varepsilon),$$

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**Theorem (2014, S. K. Kim, H. J. Lee)**

$X$ is **uniformly convex** if and only $(X, \mathbb{K})$ has the dual property.

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**Theorem (2014, S. K. Kim, H. J. Lee)**

$X$ is **uniformly convex** if and only if $(X, \mathbb{K})$ has the dual property.

**Counterexample (D., 2016)**

There are many pairs $(X, Y)$ for which this property does **not** hold.

# The dual property

The dual property is **not** possible for dimensions greater than 1!

**Theorem**

If $\dim(X), \dim(Y) > 1$, then the pair $(X, Y)$ **fails** it.

# Thank you for your attention