A kind of Bishop-Phelps-Bollobás theorem

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History & Motivation

Definition

We say that a bounded linear functional $x^* \in X^*$ **attains its norm** if there exists $x_0 \in S_X$ such that

$$|x^*(x_0)| = \|x^*\| := \sup_{x \in S_X} |x^*(x)|.$$

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(b) There exists some bounded linear functional which never attains its norm.

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$$\|x^*\|_1 = \sum_{n \in \mathbb{N}} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

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If $(\alpha_n)_{n\in\mathbb{N}} \in B_{c_0}$, there exists some $n_0 \in \mathbb{N}$ such that $|\alpha_n| < 1$ for all $n \geq n_0$. This implies that

$$|x^*((\alpha_n)_n)| = \sum_{n\in\mathbb{N}} \frac{1}{2^n}|\alpha_n| < \sum_{n\in\mathbb{N}} \frac{1}{2^n} = 1 = \|x^*\|.$$

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James Theorem (1964)

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Observation

- The completeness is necessary to get James theorem (James, 1971).

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for some $y \in S_Y$.

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Question: Is this true for bounded linear operators?

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We say that a bounded linear operator $T \in \mathcal{L}(X,Y)$ **attains its norm** if there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = \|T\| := \sup_{x \in S_X} \|T(x)\|.$$

The set of all norm attaining operators is denoted by $NA(X,Y)$.

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Given an operator $T \in \mathcal{L}(X,Y)$, it is possible to get another operator $S \in \mathcal{L}(X,Y)$

$$\|S(x_0)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon$$

for some $x_0 \in S_X$?

History & Motivation

Lindenstrauss' counterexample, 1963

There exists a Banach space X such that

$$\overline{NA(X,X)} \neq \mathcal{L}(X,X)$$

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There exists a Banach space X such that

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showing that the Bishop-Phelps result **does not** hold for bounded linear operators in general.

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Positive Results

(a) **(Lindenstrauss, 1963)** If X is reflexive, then $NA(X,Y)$ is dense in $\mathcal{L}(X,Y)$.

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(a) **(Lindenstrauss, 1963)** If X is reflexive, then $NA(X,Y)$ is dense in $\mathcal{L}(X,Y)$.

(b) **(Bourgain, 1977)** If X has the Radon-Nikodým property, then $NA(X,Y)$ is dense in $\mathcal{L}(X,Y)$.

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Open Question

$\overline{N(X, \mathbb{R}^2)} = \mathcal{L}(X, \mathbb{R}^2)$ for every Banach space X?

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Observation: This is true when X is a uniformly convex Banach space.

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The Bishop-Phelps-Bollobás theorem (1970)

Let X be a Banach space. Given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $x^* \in B_{X^*}$ and $x \in B_X$ are such that

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

there are $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(y)| = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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$$\boxed{|x^*(x)| > 1 - \eta(\varepsilon)} \longrightarrow \eta(\varepsilon) = \frac{\varepsilon^2}{2} \quad \text{(Martín, 2014)}$$

there are $y \in S_X$ and $y^* \in S_{X^*}$ such that

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By the **Bishop-Phelps-Bollobás theorem**, there exists another functional $y^* \in X^*$ and another point $y_0 \in X$ such that

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So the BPB theorem implies the BP theorem.

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In 2008, Acosta, Aron, García and Maestre introduced

The Bishop-Phelps-Bollobás property

A pair of Banach spaces (X, Y) is said to have the **BPBp** if given $\varepsilon > 0$, then there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ are such that

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- $(C_0(L), L_p(\mu))$ for every Hausdorff locally compact space L and $1 \leq p < \infty$.

Definition & Some Results

4. COLLECTION OF THE CLASSIC BANACH SPACES WHICH HAVE THE BPBp FOR OPERATORS

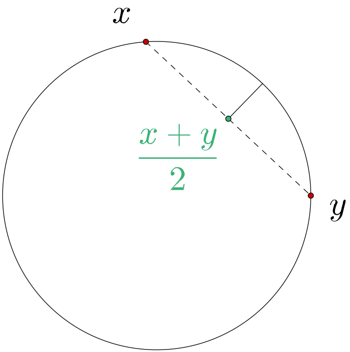
	RANGE SPACES																					
DOMAIN SPACES	FD	ℓ_1^n	ℓ_p^n	ℓ_q^n	ℓ_∞^n	c_0	ℓ_1	ℓ_p	ℓ_q	ℓ_∞	$L_1(\mu)$	$L_1(\nu)$	$L_p(\mu)$	$L_p(\nu)$	$L_q(\mu)$	$L_q(\nu)$	$L_\infty(\mu)$	$L_\infty(\nu)$	$C(K)$	$C(S)$	$C_0(S)$	$C_0(L)$
FD	✓	✓	✓	✓	✓																	
ℓ_1^n	✓	✓	✓	✓	✓		✓	✓	✓		✓	✓	✓	✓	✓	✓						
ℓ_p^n	✓	✓	✓	✓	✓		✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓				
ℓ_q^n	✓	✓	✓	✓	✓		✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓				
ℓ_∞^n	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
c_0			✓	✓				✓	✓				✓	✓	✓	✓					✓	✓
ℓ_1	✓	✓	✓	✓			✓	✓	✓		✓	✓	✓	✓	✓	✓			✓	✓		
ℓ_p		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
ℓ_q		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
ℓ_∞		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓						
$L_1(\mu)$		✓	✓	✓			✓	✓	✓		✓	✓	✓	✓								
$L_1(\nu)$		✓	✓	✓			✓	✓	✓		✓	✓	✓	✓								
$L_p(\mu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
$L_p(\nu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
$L_q(\mu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
$L_q(\nu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
$L_\infty(\mu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
$L_\infty(\nu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
$C(K)$			✓	✓				✓	✓				✓	✓	✓	✓			✓	✓		
$C(S)$			✓	✓				✓	✓				✓	✓	✓	✓			✓	✓		
$C_0(S)$		✓	✓	✓			✓	✓	✓		✓	✓	✓	✓	✓	✓					✓	✓
$C_0(L)$		✓	✓	✓			✓	✓	✓		✓	✓	✓	✓	✓	✓					✓	✓

Cuadro 1: FD = Finite-dimensional, RED = real case and BLUE = complex case

The Bishop-Phelps-Bollobás point property

A Banach space X is **uniformly convex** if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$x, y \in S_X \text{ and } \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon).$$



The Bishop-Phelps-Bollobás point property

Kim-Lee Theorem (2014)

A Banach space X is **uniformly convex** if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x \in B_X$ satisfy

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

there is $x_0 \in S_X$ such that

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Question Is it true for operators?

The Bishop-Phelps-Bollobás point property

The pair (ℓ_2^2, ℓ_2^2) **does not** satisfy this property. (D., 2016)

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A pair (X, Y) is said to have the **BPBpp** if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there exists $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ such that

$$\|S(x_0)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

The Bishop-Phelps-Bollobás point property

Definition

A Banach space X is **uniformly smooth** if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists whenever $x \in S_X$ and $y \in X$.

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Theorem

The Banach space X is uniformly smooth if and only if the pair $(X, \mathbb{K})$ has the BPBpp.

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The Banach space X is uniformly smooth if and only if the pair $(X, \mathbb{K})$ has the BPBpp.

Examples:

(a) If H is a Hilbert space, then the pair $(H, \mathbb{K})$ has the BPBpp.

(b) The pair $(L_p(\mu), \mathbb{K})$ has the BPBpp for a σ-finite measure μ and $1 < p < \infty$.

The Bishop-Phelps-Bollobás point property

Theorem

Let X be a Banach space. Suppose that there is some Banach space Y such that the pair (X, Y) has the BPBpp. Then X is uniformly smooth.

The Bishop-Phelps-Bollobás point property

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Let X be a Banach space. Suppose that there is some Banach space Y such that the pair (X, Y) has the BPBpp. Then X is uniformly smooth.

Examples:

(a) The pair (c_0, Y) **fails** the BPBpp for all Banach space Y.

(b) The pair (ℓ_1, Y) **fails** the BPBpp for all Banach space Y.

The Bishop-Phelps-Bollobás point property

Theorem

Assume that X is uniformly smooth and that Y has the property β. Then the pair (X, Y) has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

Assume that X is uniformly smooth and that Y has the property β. Then the pair (X, Y) has the BPBpp.

Examples:

(a) The pairs $(L_p(\mu), c_0)$ and $(L_p(\mu), \ell_\infty)$ have the BPBpp for a σ-finite measure μ and $1 < p < \infty$.

(b) If H is a Hilbert space, then the pairs (H, c_0) and (H, ℓ_∞) has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

Let H be a Hilbert space and let Y be any Banach space. Then the pair (H, Y) has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

Let H be a Hilbert space and let Y be any Banach space. Then the pair (H, Y) has the BPBpp.

Open problem: The pair (X, H) has the BPBpp when X is any Banach space and H is a Hilbert space?

The Bishop-Phelps-Bollobás point property

Theorem

Let X be a uniformly smooth Banach space and A be a uniform algebra. The pair (X, A) has the BPBpp.

The Bishop-Phelps-Bollobás point property

Theorem

Let X be a uniformly smooth Banach space and A be a uniform algebra. The pair (X, A) has the BPBpp.

Examples:

(a) The pair $(L_p(\mu), C(K))$ has the BPBpp.

(b) The pair $(H, C(K))$ has the BPBpp.

The Bishop-Phelps-Bollobás point property

Counterexample with X uniformly smooth

Kim and Lee proved that a 2-dimensional real Banach space X is uniformly convex if and only if the pair (X, Y) has the BPBp for all Banach spaces Y.

So, if X_0 is a 2-dimensional real Banach space which is uniformly smooth but not strictly convex, there exists a Banach space Y_0 such that the pair (X_0, Y_0) do not have the BPBp and thus it fails the BPBpp.

Open Questions

It is true or false?

(a) The set $NA(X, \mathbb{R}^2)$ is dense in $\mathcal{L}(X, \mathbb{R}^2)$ for all Banach space X?

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(a) The set $NA(X, \mathbb{R}^2)$ is dense in $\mathcal{L}(X, \mathbb{R}^2)$ for all Banach space X?

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Open Questions

It is true or false?

(a) The set $NA(X, \mathbb{R}^2)$ is dense in $\mathcal{L}(X, \mathbb{R}^2)$ for all Banach space X?

(b) If (X, Y) has the BPBpp for all Y, then X must be uniformly convex?

(c) The pair $(L_p(\mu), Y)$ has the BPBpp for all measure μ and any Banach space X?

Thank you for your attention!