Recent results of the Bishop-Phelps-Bollobás point property

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Notation

X, Y and Z are real or complex Banach spaces.

- $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$,
- B_X is the closed unit ball of X,
- S_X is the unit sphere of X,
- $\mathcal{L}(X, Y)$ continuous linear operators from X into Y,
- $K(X, Y)$ compact linear operators from X into Y,
- $X^* = \mathcal{L}(X; \mathbb{K})$ topological dual of X.

Motivation & History background

Definition

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James theorem (1957)

A Banach space X is **reflexive** if and only if every bounded linear functional is norm attaining.

Motivation & History background

Bishop-Phelps theorem (1961)

Every element in X^* can be approximated by a norm attaining linear functional. In other words, $\overline{\mathrm{NA}(X)} = X^*$.

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Question (Bishop-Phelps)

Is it true for bounded linear operators?

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We say that a bounded linear operator $T \in \mathcal{L}(X, Y)$ **attains its norm** if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\|$.

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$\text{NA}(X, Y)$ is the set of **all norm attaining operators**.

(1963, Lindenstrauss) Counterexample

There exists a Banach space X such that

$$\overline{\text{NA}(X, X)} \neq \mathcal{L}(X, X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators.

Motivation & History background

In 1970, Bollobás improved the Bishop-Phelps theorem.

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1970, Bollobás, Bishop-Phelps-Bollobás theorem
(2014, M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido)

Let $\varepsilon \in (0, 2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = |y^*(y)| = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

Motivation & History background

Observation 1

Bishop-Phelps-Bollobás theorem $\Rightarrow$ Bishop-Phelps theorem.

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Bishop-Phelps-Bollobás theorem $\Rightarrow$ Bishop-Phelps theorem.

Observation 2

It is **not** expected that there exists a Bishop-Phelps-Bollobás theorem version for bounded linear operators in general.

Motivation & History background

(2008, M. Acosta, R. Aron, D. García, M. Maestre)

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Bishop-Phelps-Bollobás property (BPBp)

A pair of Banach spaces (X, Y) is said to have the **BPBp** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exist $S \in \mathcal{L}(X,Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \;\text{ and }\; \|T - S\| < \varepsilon.$$

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- Y is uniformly convex.
- $Y = C(K)$ for K a compact Haurdorff space.

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  (2014, Y. S. Choi, S. K. Kim, H. J. Lee, M. Martín)
- (X, Y) has the BPBp whenever X uniformly convex.
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- (X, Y) has the BPBp whenever X uniformly convex.
  (2014, S. K. Kim, H. J. Lee)
- $(C(K), L_1(\mu))$ has the BPBp.
  (2016, M. Acosta)

The Bishop-Phelps-Bollobás point property

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Bishop-Phelps-Bollobás **point** property (BPBpp)

A pair of Banach spaces (X, Y) is said to have the **BPBpp** if for every $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

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It is clear that BPBpp $\Rightarrow$ BPBp.

First results about the BPBpp

(2016, D., S. K. Kim and H. J. Lee)

- $(X, \mathbb{K})$ has the BPBpp if and only if X is uniformly smooth.

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- (H, Y) has the BPBpp for all Hilbert spaces H and any Y.
- (X, Y) has the BPBpp for X uniformly smooth and Y property β.
- there are uniformly smooth Banach spaces X such that the pair (X, Y) **fails** the BPBpp for some Y.

Recent results

Recent results about the BPBpp

Stability results

Proposition

Let X_1 be a one-complemented subspace of X. If (X, Y) has the BPBpp, then (X_1, Y) has the BPBpp.

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Questions

(a) Is this true for the BPBp? (201?, D., García, Maestre, Martín)

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(a) Is this true for the BPBp? (201?, D., García, Maestre, Martín)

(b) Is this true for norm attaining operators?

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Stability results

Proposition ((201?, D., García, Maestre, Martín) adapted)

If $Y = Y_1 \oplus_a Y_2$ and (X, Y) has the BPBpp, then (X, Y_j) has the BPBpp.

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Stability results

Proposition ((201?, D., García, Maestre, Martín) adapted)

If $Y = Y_1 \oplus_a Y_2$ and (X, Y) has the BPBpp, then (X, Y_j) has the BPBpp.

Proposition ((2015, Aron, Choi, Kim, Lee, Martín) adapted)

If $(X, C(K, Y))$ has the BPBpp, then (X, Y) has the BPBpp.

Recent results about the BPBpp

Universal properties

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Definition (2014, Aron, Choi, Kim, Lee, Martín)

(a) X is **universal BPBpp domain space** if (X, Y) has the BPBpp for all Y.

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- Hilbert spaces are universal BPBpp domain spaces.

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Examples (2016, D., S. K. Kim, H. J. Lee)

- Hilbert spaces are universal BPBpp domain spaces.
- Uniform algebras and Banach spaces with property β are universal BPBpp range spaces.

Recent results about the BPBpp

Universal properties

Question

We know that Hilbert spaces are universal BPBpp domain spaces.

Recent results about the BPBpp

Universal properties

Question

We know that Hilbert spaces are universal BPBpp domain spaces.

Is it possible to extend the result for L_p-spaces with $1 < p < \infty$?

Recent results about the BPBpp

Universal properties

Theorem

If X is universal BPBpp domain space, then X is uniformly convex.

Recent results about the BPBpp

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Theorem

If X is universal BPBpp domain space and X is isomorphic to a Hilbert space, then $\delta_X(\varepsilon) \geq C\varepsilon^2$.

Recent results about the BPBpp

Universal properties

Corollary

$L_p(\mu)$ is **not** a BPBpp domain space for $p > 2$.

Recent results about the BPBpp

Universal properties

Corollary

$L_p(\mu)$ is **not** a BPBpp domain space for $p > 2$.

Question

Is $L_p(\mu)$ a BPBpp domain space for $1 < p < 2$?

Recent results about the BPBpp

Universal properties

ACK_ρ-structure (2017, Cascales, Guirao, Kadets, Soloviova)

Theorem

If Y has ACK_ρ-structure, then Y is universal BPBpp range space.

Recent results about the BPBpp

Universal properties

ACK_ρ-structure (2017, Cascales, Guirao, Kadets, Soloviova)

Theorem

If Y has ACK_ρ-structure, then Y is universal BPBpp range space.

- $C(K)$ and $C_0(L)$ and, more in general, uniform algebras.
- Banach spaces with property β.
- finite ℓ_∞-sums of Banach spaces with ACK_ρ-structure.
- $c_0(Y)$, $\ell_\infty(Y)$ when Y has ACK_ρ-structure.
- $C(K, Y)$ when Y has ACK_ρ-structure.

Recent results about the BPBpp

Universal properties

Counterexample

For $p \geq 2$, there is a Banach space X_p which is **uniformly convex and uniformly smooth** such that (X_p, ℓ_p^2) **fails** the BPBpp.

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For $p \geq 2$, there is a Banach space X_p which is **uniformly convex and uniformly smooth** such that (X_p, ℓ_p^2) **fails** the BPBpp.

- Note that (X_p, ℓ_p^2) has the BPBp since X_p is uniformly convex.

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(1) If Y is universal BPBp range space, then Y is universal BPBpp range for uniformly smooth X?

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- Note that (X_p, ℓ_p^2) has the BPBp since X_p is uniformly convex.

Questions

(1) If Y is universal BPBp range space, then Y is universal BPBpp range for uniformly smooth X?

(2) It is not known whether all finite dimensional spaces are universal **BPBp** range spaces or even if they have Lindenstrauss property B.

Recent results about the BPBpp

The BPBpp for compact operators

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BPBpp for compact operators

A pair of Banach spaces (X, Y) is said to have the **BPBpp for compact operators** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in K(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exists $S \in K(X, Y)$ with $\|S\| = 1$ such that

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The BPBpp for compact operators

- (H, Y) for H Hilbert spaces and any Y.

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- (H, Y) for H Hilbert spaces and any Y.
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(2017, D., García, Maestre, Martín)

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- $(X, \ell_p(Y)) \Rightarrow (X, L_p(\mu, Y))$ for $1 \leq p < \infty$.

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- $(X, \ell_p(Y)) \Rightarrow (X, L_p(\mu, Y))$ for $1 \leq p < \infty$.
- $(X, Y) \Rightarrow (X, L_\infty(\mu, Y))$
- $(X, Y) \Rightarrow (X, C(K, Y))$.

The dual property

The dual property

Recall that (X, Y) has the **BPBpp** if $\forall\ \varepsilon \in (0,1)$, $\exists\ \eta(\varepsilon) > 0$:

$$T \in S_{\mathcal{L}(X,Y)}, x \in S_X \text{ with } \|T(x)\| > 1 - \eta(\varepsilon),$$

$\Rightarrow \exists\ S \in S_{\mathcal{L}(X,Y)}$ with

$$\|S(x)\| = 1 \quad \text{and} \quad \|T - S\| < \varepsilon.$$

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A possible dual property: (2016, D.)

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Theorem (2014, S. K. Kim, H. J. Lee)

X is **uniformly convex** if and only if $(X, \mathbb{K})$ has the dual property.

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Theorem (2014, S. K. Kim, H. J. Lee)

X is **uniformly convex** if and only if $(X, \mathbb{K})$ has the dual property.

Counterexample (D., 2016)

There are many pairs (X, Y) for which this property does **not** hold.

The dual property

The dual property is **not** possible for dimensions greater than 1!

Theorem

If $\dim(X), \dim(Y) > 1$, then the pair (X, Y) **fails** it.

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Theorem

If $\dim(X), \dim(Y) > 1$, then the pair (X, Y) **fails** it.

Proof.

- Reducing the proof for 2-dimensional spaces.
- Dividing the proof in two cases:
  - X is Hilbert (John's maximal ellipsoid theorem)
  - X is not Hilbert (Day's and Nordlander's theorems)

□

Thank you for your attention