Dense subspaces which admit smooth norms

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(ii) *If the dual norm is Fréchet differentiable, then* X *is reflexive.*

(iii) *If the dual norm on* X^* *is LUR, then* $\|\cdot\|$ *is Fréchet differentiable.*

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Question *Does every Asplund Banach space admit a C^1-smooth bump function?*

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- (J. Vanderwerff, 1992) *If X is a separable Banach space and L is a subspace of dimensional $\aleph_0$, then X admits an equivalent LUR norm which is Fréchet differentiable on $L \setminus \{0\}$. In particular, any* **normed** *space of dimension $\aleph_0$ admits a Fréchet differentiable norm.*

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Let Γ *be an uncountable set and* $\mathcal{F}$ *be a normed space of all finitely supported vectors in* $\ell_1(\Gamma)$ *endowed with the* ℓ_1*-norm. Does* $\mathcal{F}$ *admit a Fréchet smooth norm?*

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Q3. What can one says about the whole space X if there exists a dense subspace Y which admits a C^k-smooth norm?

The results

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Given a normed space $(X, \|\cdot\|)$ and $\varepsilon > 0$, we say that a new norm $|||\cdot|||$ approximates the original one $\|\cdot\|$ if

$$|||x||| \leq \|x\| \leq (1+\varepsilon)|||x|||$$

for all $x \in X$.

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Then, the Minkowski functional μ *on* B *is an equivalent* C^k*-smooth norm on* X*.*

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Theorem 1: *The space ℓ_∞^F admits an analytic norm which approximates the original one.*

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Corollary 2: *Let X be a separable Banach space. Then, there is a dense subspace Y of X which admits an analytic norm and approximates the original one.*

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Corollary 2: *Let X be a separable Banach space. Then, there is a dense subspace Y of X which admits an analytic norm and approximates the original one.*

Corollary 3: *The normed space $\mathcal{F}$ of all finitely supported vectors in $\ell_1(c)$, where c denotes a set of cardinality continuum, endowed with the ℓ_1-norm, admits an equivalent analytic norm which approximates the original one.*

The results

Theorem 4: *Let X be a Banach space with a suppression 1-unconditional Schauder basis $\{e_\gamma\}_{\gamma \in \Gamma}$ and set $Y := \text{span}\{e_\gamma\}_{\gamma \in \Gamma}$. Then, Y is a dense subspace of X which admits a C^∞-smooth norm and approximates the original one.*

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GENERAL QUESTION

How different can two dense subspaces of a Banach space be?

Thank you
for your attention