Octahedral norms in free Banach lattices (Part II)

Sheldon Dantas

Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Mathematics

Research supported by

Ministry of Education, Youth, and Sports of the Czech Republic

Estonian Research Council

November 13th, 2020
Seminars on Functional Analysis, Tartu University

Joint work with
G. Martínez-Cervantes, J.D. Rodríguez Abellán, and A. Rueda Zoca

Joint work with G. Martínez-Cervantes, J.D. Rodríguez Abellán, and A. Rueda Zoca

Schedule

Joint work with G. Martínez-Cervantes, J.D. Rodríguez Abellán, and A. Rueda Zoca

Schedule

- Background

Joint work with G. Martínez-Cervantes, J.D. Rodríguez Abellán, and A. Rueda Zoca

Schedule

- Background
- Almost Squareness

Joint work with G. Martínez-Cervantes, J.D. Rodríguez Abellán, and A. Rueda Zoca

Schedule

- Background
- Almost Squareness
- Cunningham Algebras

Joint work with G. Martínez-Cervantes, J.D. Rodríguez Abellán, and A. Rueda Zoca

Schedule

- Background
- Almost Squareness
- Cunningham Algebras
- Open Questions

Motivation and Background

Let E be a real Banach space.

Let E be a real Banach space.

Definition

A **slice** of the unit ball B_E is a set of the form

$$S(B_E, f, \alpha) := \{x \in B_E : f(x) > 1 - \alpha\},$$

where $f \in B_{E^*}$ and $\alpha > 0$.

Let E be a real Banach space.

Definition

A **slice** of the unit ball B_E is a set of the form

$$S(B_E, f, \alpha) := \{x \in B_E : f(x) > 1 - \alpha\},$$

where $f \in B_{E^*}$ and $\alpha > 0$.

If E is a dual space, say $E = F^*$, by a **w^*-slice** of B_{E^*}, we mean a slice of the form $S(B_E, y, \alpha)$, where $y \in F$.

Definition (G. Godefroy, 1989)

The norm of a Banach space E is said to be **octahedral** if, for every finite-dimensional subspace Z of E and every $\varepsilon > 0$, there exists $x \in S_E$ such that

$$\|z + \lambda x\| \geq (1 - \varepsilon)(\|z\| + |\lambda|)$$

for every $z \in Z$ and every $\lambda \in \mathbb{R}$.

Definition (G. Godefroy, 1989)

The norm of a Banach space E is said to be **octahedral** if, for every finite-dimensional subspace Z of E and every $\varepsilon > 0$, there exists $x \in S_E$ such that

$$\|z + \lambda x\| \geq (1 - \varepsilon)(\|z\| + |\lambda|)$$

for every $z \in Z$ and every $\lambda \in \mathbb{R}$.

(R. Haller, J. Langemets and M. Põldvere, 2015)

A Banach space E has an octahedral norm if and only if given a finite collection $x_1, \ldots, x_n \in S_E$ and $\varepsilon > 0$, we can find $x \in S_E$ such that

$$\|x_i + x\| > 2 - \varepsilon,$$

for every $i \in \{1, \ldots, n\}$.

SD2P

We say that E satisfies the strong diameter two property (SD2P, for short) if every convex combination of slices of the closed unit ball of E has diameter two.

SD2P

We say that E satisfies the strong diameter two property (SD2P, for short) if every convex combination of slices of the closed unit ball of E has diameter two.

w^*-SD2P

If E is itself a dual Banach space, then we can define the w^*-strong diameter two property (w^*-SD2P, for short) by using convex combinations of w^*-slices of the unit ball of E.

Examples of Banach spaces satisfying the SD2P

- Infinite dimensional uniform algebras.
  (O. Nygaard, D. Werner, 2001)

Examples of Banach spaces satisfying the SD2P

- Infinite dimensional uniform algebras.
  (O. Nygaard, D. Werner, 2001)
- Banach spaces satisfying the DP.
  (R.V. Shvydkoy, 2000)

Examples of Banach spaces satisfying the SD2P

- Infinite dimensional uniform algebras.
  (O. Nygaard, D. Werner, 2001)
- Banach spaces satisfying the DP.
  (R.V. Shvydkoy, 2000)
- Non-reflexive M-embedded spaces.
  (G. López-Pérez)

Examples of Banach spaces satisfying the SD2P

- Infinite dimensional uniform algebras.
  (O. Nygaard, D. Werner, 2001)
- Banach spaces satisfying the DP.
  (R.V. Shvydkoy, 2000)
- Non-reflexive M-embedded spaces.
  (G. López-Pérez)

(J. Becerra Guerrero, G. López-Pérez, A. Rueda Zoca, 2014)

Let E be a Banach space. The following are equivalent.

(a) The norm of E is octahedral.

(b) Every convex combination of w^*-slices in B_{E^*} has diameter 2.

FBL *vs* $\widehat{\otimes}_{\pi}$

FBL *vs* $\widehat{\otimes}_\pi$

- The difficulty of dealing with the norm

FBL *vs* $\widehat{\otimes}_\pi$

- The difficulty of dealing with the norm
- Transfer the techniques

FBL *vs* $\widehat{\otimes}_\pi$

(J. Langemets, V. Lima, A. Rueda Zoca, RACSAM, 2017)

Let E and F be Banach spaces. Assume that E is ASQ and F is Asplund. If either E^* or F^* has the AP, then the norm of $E^* \widehat{\otimes}_\pi F^*$ is octahedral.

FBL *vs* $\widehat{\otimes}_\pi$

(J. Langemets, V. Lima, A. Rueda Zoca, RACSAM, 2017)

Let E and F be Banach spaces. Assume that E is ASQ and F is Asplund. If either E^* or F^* has the AP, then the norm of $E^*\widehat{\otimes}_\pi F^*$ is octahedral.

(J. Langemets, V. Lima, A. Rueda Zoca, Q.J. Math., 2017)

Let E be a non-reflexive L-embedded space and F be a Banach space. If either E^{**} or F has the MAP, then the norm of $E\widehat{\otimes}_\pi F$ is octahedral.

FBL *vs* $\widehat{\otimes}_\pi$

(J. Langemets, V. Lima, A. Rueda Zoca, RACSAM, 2017)

Let E and F be Banach spaces. Assume that E is ASQ and F is Asplund. If either E^* or F^* has the AP, then the norm of $E^*\widehat{\otimes}_\pi F^*$ is octahedral.

(J. Langemets, V. Lima, A. Rueda Zoca, Q.J. Math., 2017)

Let E be a non-reflexive L-embedded space and F be a Banach space. If either E^{**} or F has the MAP, then the norm of $E\widehat{\otimes}_\pi F$ is octahedral.

(J. Langemets, V. Lima, A. Rueda Zoca, Q.J. Math., 2017)

Let E and F be Banach spaces. Assume that F is finite dimensional and that the norm of $E\widehat{\otimes}_\pi F$ is octahedral. Then, the norm of E is octahedral.

Almost square Banach spaces

ASQ (T.A. Abrahamsen, J. Langemets, and V. Lima, 2016)

A Banach space E is **almost square** (ASQ, for short) if, for every $\{x_1, \ldots, x_n\} \subset S_E$ and every $\varepsilon > 0$, there exists a sequence $(y_k) \subset S_E$ such that

$$\|x_i \pm y_k\| \longrightarrow 1$$

for every $i = 1, \ldots, n$.

ASQ (T.A. Abrahamsen, J. Langemets, and V. Lima, 2016)

A Banach space E is **almost square** (ASQ, for short) if, for every $\{x_1, \ldots, x_n\} \subset S_E$ and every $\varepsilon > 0$, there exists a sequence $(y_k) \subset S_E$ such that

$$\|x_i \pm y_k\| \longrightarrow 1$$

for every $i = 1, \ldots, n$.

ASQ

- ASQ spaces satisfy the SD2P.

ASQ (T.A. Abrahamsen, J. Langemets, and V. Lima, 2016)

A Banach space E is **almost square** (ASQ, for short) if, for every $\{x_1, \ldots, x_n\} \subset S_E$ and every $\varepsilon > 0$, there exists a sequence $(y_k) \subset S_E$ such that

$$\|x_i \pm y_k\| \longrightarrow 1$$

for every $i = 1, \ldots, n$.

ASQ

- ASQ spaces satisfy the SD2P.
- ASQ contain an isomorphic copy of c_0.
  (T.A. Abrahamsen, J. Langemets, and V. Lima, 2016)

ASQ (T.A. Abrahamsen, J. Langemets, and V. Lima, 2016)

A Banach space E is **almost square** (ASQ, for short) if, for every $\{x_1, \ldots, x_n\} \subset S_E$ and every $\varepsilon > 0$, there exists a sequence $(y_k) \subset S_E$ such that

$$\|x_i \pm y_k\| \longrightarrow 1$$

for every $i = 1, \ldots, n$.

ASQ

- ASQ spaces satisfy the SD2P.
- ASQ contain an isomorphic copy of c_0.
  (T.A. Abrahamsen, J. Langemets, and V. Lima, 2016)
- If a Banach space E satisfies that E^* contains an isomorphic copy of c_0, then there exists an equivalent renorming of E, say F, such that F^* is ASQ.
  (T.A. Abrahamsen, P. Hájek, S. Troyanski, 2020)

Theorem

Let E be a Banach space. If the dual E^* is ASQ, then then norm of $FBL[E]$ is octahedral.

CUNNINGHAM ALGEBRAS

Cunningham Algebras

Given a Banach space E, we denote by $C(E)$ the **Cunningham algebra** of E and by $E^{(\infty}$ the completion of the normed space $\bigcup_{n=0}^{\infty} E^{(2n}$, where $(E^{(2n})_{n=0}^{\infty}$ is the sequence of even duals such that $E \subseteq E^{**} \subseteq E^{(4} \subset \ldots \subseteq E^{(2n} \subseteq \ldots$.

Cunningham Algebras

Given a Banach space E, we denote by $C(E)$ the **Cunningham algebra** of E and by $E^{(\infty}$ the completion of the normed space $\bigcup_{n=0}^{\infty} E^{(2n}$, where $(E^{(2n})_{n=0}^{\infty}$ is the sequence of even duals such that $E \subseteq E^{**} \subseteq E^{(4} \subset \ldots \subseteq E^{(2n} \subseteq \ldots$.

Theorem

Let E be a Banach space and suppose that $C(E^{(\infty})$ is infinite-dimensional. Then, given $f_1, \ldots, f_n \in S_{FBL[E]}$ and $\varepsilon > 0$, there exists an element $x \in B_E$ such that

$$\|f_i + \delta_x\|_{FBL[E]} > 2 - \varepsilon$$

holds for every $i \in \{1, \ldots, n\}$. In particular, the norm of $FBL[E]$ is octahedral.

Infinite-dimensional $C(E^{(\infty)})$

$C(E^{(\infty)})$ is infinite-dimensional if the Banach space E satisfies

Infinite-dimensional $C(E^{(\infty})$

$C(E^{(\infty})$ is infinite-dimensional if the Banach space E satisfies

- E is a non-reflexive L-embedded Banach space

Infinite-dimensional $C(E^{(\infty)})$

$C(E^{(\infty)})$ is infinite-dimensional if the Banach space E satisfies

- E is a non-reflexive L-embedded Banach space
- $E = E_1 \widehat{\otimes}_\pi E_2$ under some hypothesis.
  (M.D. Acosta, J. Becerra Guerrero, 2010)

Let us recall that a Banach space E is said to be ***L*-embedded** if $E^{**} = E \oplus_1 Z$ for some subspace Z of E^{**}.

Examples of L-embedded Banach spaces

- $L_1(\mu)$-spaces
- Preduals of von Neumann algebras
- Preduals of real or complex JBW^*-triples
- Duals of M-embedded Banach spaces
- The disk algebra

(P. Harmand, D. Werner, W. Werner, 1993)
(J. Becerra Guerrero, G. López-Pérez, A.M. Peralta, A.M., Rodríguez-Palacios, 2004)

Therefore,

The norm of $FBL[E]$ is octahedral if the Banach space E is:

- an $L_1(\mu)$-space.
- a predual of von Neumann algebra.
- a predual of real or complex JBW^*-triples.
- a dual of M-embedded Banach space
- the disk algebra.
- the projective product (under some hypothesis).

Remarks and Open Questions

A Banach space E has the **Daugavet property** if every rank-one operator $T : E \to E$ satisfies the equality

$$\| T + \mathrm{Id} \| = 1 + \| T \|.$$

A Banach space E has the **Daugavet property** if every rank-one operator $T : E \to E$ satisfies the equality

$$\| T + \mathrm{Id} \| = 1 + \| T \|.$$

Examples of Banach spaces satisfying the DP

- $\mathcal{C}(K)$ for a compact Hausdorff perfect topological space K.
- $L_1(\mu)$ and $L_\infty(\mu)$ for non-atomic measures μ.
- $Lip(M)$ over a metrically convex space M.
(V. Kadets, Y. Ivakhno, D. Werner, R. Shvidkoy, G. Sirotkin)

A Banach space E has the **Daugavet property** if every rank-one operator $T : E \to E$ satisfies the equality

$$\| T + \mathrm{Id} \| = 1 + \| T \|.$$

Examples of Banach spaces satisfying the DP

- $\mathcal{C}(K)$ for a compact Hausdorff perfect topological space K.
- $L_1(\mu)$ and $L_\infty(\mu)$ for non-atomic measures μ.
- $Lip(M)$ over a metrically convex space M.
(V. Kadets, Y. Ivakhno, D. Werner, R. Shvidkoy, G. Sirotkin)

(V. Kadets, R. Shvidkoy, G. Sirotkin, D. Werner, 2007)

If E has the Daugavet property, then the norms of E and E^* are octahedral.

(A. Rueda Zoca, P. Tradacete, I. Villanueva, 2019)

If E and F are L_1-preduals with the Daugavet property, then $E\widehat{\otimes}_\pi F$ has the Daugavet property.

(A. Rueda Zoca, P. Tradacete, I. Villanueva, 2019)

If E and F are L_1-preduals with the Daugavet property, then $E \widehat{\otimes}_\pi F$ has the Daugavet property.

(M. Martín, A. Rueda Zoca, 2020)

If E is an L_1-predual with the Daugavet property, then $\widehat{\otimes}_{\pi,s,N} E$ has the Daugavet property.

Open Questions

(1) If E is an L_1-predual with the DP then $FBL[E]$ is octahedral?

Open Questions

(1) If E is an L_1-predual with the DP then $FBL[E]$ is octahedral?

Proposition

Let E be an L_1-predual with octahedral norm and denote by $S^+_{FBL[E]}$ the positive elements of $S_{FBL[E]}$. Then, for every $f_1, \dots, f_n \in S^+_{FBL[E]}$ and every $\varepsilon > 0$, there exists $x \in S_E$ such that

$$\| f_i \pm |\delta_x| \| > 2 - \varepsilon.$$

Open Questions

(1) If E is an L_1-predual with the DP then $FBL[E]$ is octahedral?

(2) Does $FBL[E]$ have the DP for every Banach space E?

Open Questions

(1) If E is an L_1-predual with the DP then $FBL[E]$ is octahedral?

(2) Does $FBL[E]$ have the DP for every Banach space E?

(3) Are there free Banach lattices which do not have octahedral norms?

Open Questions

(1) If E is an L_1-predual with the DP then $FBL[E]$ is octahedral?

(2) Does $FBL[E]$ have the DP for every Banach space E?

(3) Are there free Banach lattices which do not have octahedral norms? In fact,

  (3.1) Let E be a 1-dimensional space. Is $FBL[E]$ octahedral?

Open Questions

(1) If E is an L_1-predual with the DP then $FBL[E]$ is octahedral?

(2) Does $FBL[E]$ have the DP for every Banach space E?

(3) Are there free Banach lattices which do not have octahedral norms? In fact,

  (3.1) Let E be a 1-dimensional space. Is $FBL[E]$ octahedral? No!

Open Questions

(1) If E is an L_1-predual with the DP then $FBL[E]$ is octahedral?

(2) Does $FBL[E]$ have the DP for every Banach space E?

(3) Are there free Banach lattices which do not have octahedral norms? In fact,

- (3.1) Let E be a 1-dimensional space. Is $FBL[E]$ octahedral? No!
- (3.2) Let E be a finite dimensional space with $\dim(E) \geq 2$. Is $FBL[E]$ octahedral?

Open Questions

(1) If E is an L_1-predual with the DP then $FBL[E]$ is octahedral?

(2) Does $FBL[E]$ have the DP for every Banach space E?

(3) If $\dim(E) \geq 2$ then the norm of $FBL[E]$ is octahedral?

Open Questions

(3) If $\dim(E) \geq 2$ then the norm of $FBL[E]$ is octahedral?

Open Questions

(3) If $\dim(E) \geq 2$ then the norm of $FBL[E]$ is octahedral?

Proposition

Let E be a finite dimensional space with dimension $n \geq 2$. Let β be the Banach-Mazur distance between ℓ_1^n and E. Set $\alpha := \frac{2}{ne^{\beta}}$. Then, every convex combination of w^*-slices of $B_{FBL[E]^*}$ has diameter greater or equal than α.

Open Questions

(3) If $\dim(E) \geq 2$ then the norm of $FBL[E]$ is octahedral?

Proposition

Let E be a finite dimensional space with dimension $n \geq 2$. Let β be the Banach-Mazur distance between ℓ_1^n and E. Set $\alpha := \frac{2}{ne^\beta}$. Then, every convex combination of w^*-slices of $B_{FBL[E]^*}$ has diameter greater or equal than α.

Corollary

Let E be a finite dimensional space with dimension $n \geq 2$. Then, the norm of $FBL[E]$ is rough. In particular, it is nowhere Fréchet differentiable.

Thank You
For Your Attention