Octahedral norms in free Banach lattices

PART I

Seminars on Functional Analysis, Tartu University

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In particular, $\|x\| = \||x|\|$.

Example

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- If X is a non-empty set, the set $\mathbb{R}^X := \{f : X \longrightarrow \mathbb{R}\}$ with the order given by $f \leq g$ if, and only if, $f(x) \leq g(x)$ for every $x \in X$ is a vector lattice.

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- ℓ_p and $L_p(\mu)$ are Banach lattices for every $1 \leq p \leq \infty$.

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- Y is an ideal if moreover, if $f \in Y$ and $|g| \leq |f|$ then $g \in Y$. This makes X/Y a Banach lattice.

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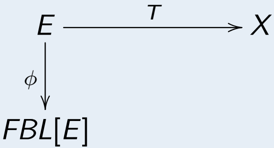

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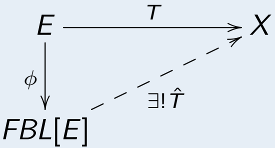

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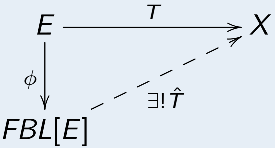

It exists and is unique up to isometries.

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$$\|f\|_{FBL[E]} = \sup\left\{ \sum_{i=1}^{n} |f(x_i^*)| : n \in \mathbb{N}, x_i^* \in E^*, \ \sup_{x \in B_E} \sum_{i=1}^{n} |x_i^*(x)| \leq 1 \right\}$$

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Theorem (Avilés, Rodríguez, Tradacete 2018)

The free Banach lattice generated by a Banach space E is the Banach lattice generated by $\{\delta_x : x \in E\}$ in $H_0[E]$.

Understanding the description of $FBL[E]$

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- Since $\|f\|_{FBL[E]} \geq \|f|_{B_{E^*}}\|_\infty$, we conclude that every function in $FBL[E]$ is weak*-continuous when restricted to the closed unit ball B_{E^*}.

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A function $f\colon E^* \longrightarrow \mathbb{R}$ is said to depend on finitely many coordinates $x_1, \ldots, x_n \in E$ if $f(x^*) = f(y^*)$ whenever $x^*(x_i) = y^*(x_i)$ for every $i \leq n$.

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Recall that any function $f\colon E^* \longrightarrow \mathbb{R}$ is positively homogeneous (i.e. $f(\lambda x^*)) = \lambda f(x^*)$ for every $\lambda \ge 0$).

Lemma [A. Avilés, G.M.C., J. Rodríguez, P. Tradacete, 2020]

Let E be a Banach space and $f : E^* \to \mathbb{R}$ be a positively homogeneous function such that:

(i) $f|_{B_{E^*}}$ is norm-continuous;

(ii) f depends on finitely many coordinates of E.

Then $f \in FBL[E]$.

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2. If a positively homogeneous function $f : E^* \to \mathbb{R}$ depends on finitely many coordinates and its restriction to B_{E^*} is norm-continuous, then f belongs to $FBL[E]$;
3. The family of functions satisfying the conditions of *(2)* is a dense (nonclosed) **sublattice** of $FBL[E]$.

Examples.

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Lemma [B. de Pagter, A.W. Wickstead / A. Avilés, J. Rodríguez, P. Tradacete]

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It follows that the set

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Moreover, A is symmetric and convex. Thus

$$\overline{A}^{w^*} = \overline{conv}^{w^*}(A) = B_{FBL[E]^*}.$$

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In particular, $FBL[E]$ is isometric to a sublattice of $FBL[E^{**}]$.

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THANK YOU FOR YOUR ATTENTION