CALCULUS 1 (BE5B01MA1)
FINAL EXAM 1 (2020, JAN, 21ST)

⋆ *This exam has* **6 problems** *with a total of* **100 points**.
⋆ *Grades classification:* **F** (<49pts), **E** (50-59), **D** (60-69), **C** (70-79), **B** (80-89), **A** (90-100).

NAME: ____________________________________________

**Exercise 1.** [25pts] Consider the function $f(x) = \dfrac{2x^2}{x^2-1}$.

(a) [1pt] Determine the domain of $f$.

(b) [2pts] Prove that $f$ is an even function.

(c) [3pts] Evaluate $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

(d) Compute

(i) [1pt] $\lim\limits_{x\to 1^+} f(x)$.

(ii) [1pt] $\lim\limits_{x\to 1^-} f(x)$.

(iii) [1pt] $\lim\limits_{x\to -1^+} f(x)$.

(iv) [1pt] $\lim\limits_{x\to -1^-} f(x)$.

(e) [2.5pts] Find $f'(x)$ and the intervals where $f$ is increasing and decreasing.

(f) [2.5pts] Find $f''(x)$ and the intervals where $f$ is concave upward and downward.

(g) [2.5pts] Find the critical point(s) of $f$.

(h) [7.5pts] Use all the previous information to sketch the graph of $f$.

**Exercise 2.** [25pts] Decide if the following statements are **true** or **false**.

- If it is **true**, explain why.
- If it is **false**, give an example that disproves the statement.

(a) [5pts] The integral $\displaystyle\int_0^{\infty} e^{-x^2}dx$ is convergent.

(b) [5pts] The integral $\displaystyle\int_1^{\infty} \frac{1+e^{-x}}{x}dx$ is divergent.

(c) [5pts] The series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ is divergent.

(d) [5pts] The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p \leqslant 0$.

(e) [5pts] The series $\displaystyle\sum_{n=1}^{\infty} \frac{2020}{2n^2+4n+3}$ is convergent.

**Exercise 3.** [15pts] Calculate the following integrals.

(a) [5pts] $\int x\sin(x)dx$

(b) [5pts] $\int \frac{2x^2 - x + 4}{x^3 + 4x}dx$

(c) [5pts] $\int_0^{2\pi} \sin^2(x)dx$

**Exercise 4.** [10pts]

(a) [1.5pts] Find the Maclaurin series of $f(x) = e^x$.

(b) [3.5pts] Find the radius of convergence of the Maclaurin series of $f(x) = e^x$.

(c) [5pts] Evaluate $\int e^{-x^2}dx$ as an infinite series.

**Exercise 5.** [15pts] Consider the parabolas $y_1 = x^2$ and $y_2 = 2x - x^2$.

(a) [1pts] Sketch the curves $y_1$ and $y_2$ *in the same screen*.

(b) [4pts] Find the points of intersection of the parabolas.

(c) [10pts] Find the area of the region enclosed by the parabolas $y_1$ and $y_2$.

**Exercise 6.** [10pts] Give the definition of

(a) [2pts] a *continuous function* at a point $a$.

(b) [2pts] a *differentiable function* at a point $a$.

(c) [2pts] an *integrable function* on an interval $[a, b]$.

(d) [2pts] a *convergent sequence* $(a_n)_{n=1}^{\infty}$.

(e) [2pts] a *convergent series* $\sum_{n=1}^{\infty} a_n$.

*Solutions*

**Exercise 1**:

(a) $D(f) = \{x \in \mathbb{R} : x \neq \pm 1\}$.

(b) For every $x \in \mathbb{R}$ with $x \neq \pm 1$, we have that

$$f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2 - 1} = f(x).$$

So, $f$ is an even function.

(c) We have that

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2x^2}{x^2\left(1 - \frac{1}{x^2}\right)} = \lim_{x \to \pm\infty} \frac{2}{1 - \frac{1}{x^2}} = 2.$$

(d) Computing the one-hand side limits, we have that

(i) $\lim_{x \to 1^+} f(x) = +\infty$.

(ii) $\lim_{x \to 1^-} f(x) = -\infty$.

(iii) $\lim_{x \to -1^+} f(x) = -\infty$.

(iv) $\lim_{x \to -1^-} f(x) = +\infty$.

(e) Applying the quotient rule, we get that

$$f'(x) = \frac{-4x}{(x^2 - 1)^2}.$$

Notice that the denominator is always positive. Then, $f'(x) > 0$ when $x < 0$ and $x \neq -1$ (remember that $-1$ is **not** in the domain of $f$!!!) and $f'(x) < 0$ when $x > 0$ and $x \neq 1$ (remember that 1 is **not** in the domain of $f$!!!). Therefore, $f$ is increasing on $(-\infty, -1) \cup (-1, 0)$ and decreasing on $(0, 1) \cup (1, +\infty)$.

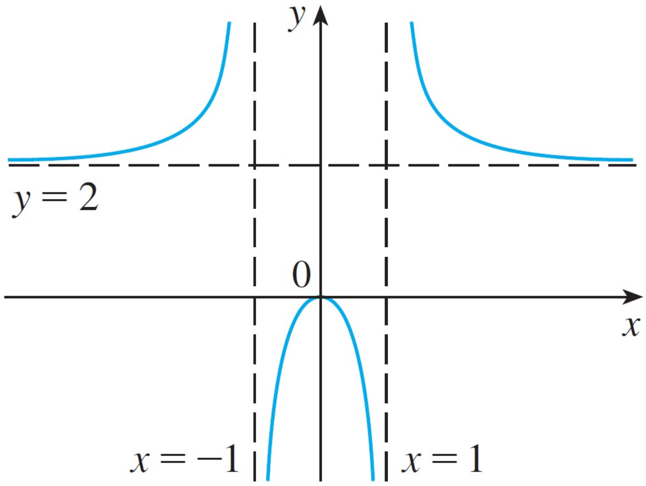
(f) Applying again the quotient rule for $f'$, we get that

$$f''(x) = \frac{12x^2 + 4}{(x^2 - 1)^3}.$$

Notice that $12x^2 + 4$ is always positive. So, $f''(x) > 0$ when $x^2 - 1 > 0$. Then, $f''(x) > 0$ when $|x| > 1$ and $f''(x) < 0$ when $|x| < 1$. Therefore, the curve is concave upward on $(-\infty, -1) \cup (1, +\infty)$ and downward on $(-1, 1)$.

(g) $f$ has no critical points since 1 and $-1$ are **not** in the domain of $f$.

(h)




**Exercise 2**:

(a) (V) Notice that

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx.$$

Now, for $x \geqslant 1$, we have that $x^2 \geqslant x$ and so $-x^2 \leqslant -x$. Since exponential is an increasing function, we get that $e^{-x^2} \leqslant e^{-x}$. Thus,

$$\int_1^\infty e^{-x^2} dx \leqslant \int_1^\infty e^{-x} dx.$$

Moreover

$$\int_1^\infty e^{-x} dx = \lim_{t\to\infty} e^{-x} dx = \lim_{t\to\infty} [-e^{-x}]_1^t = e^{-1} < \infty.$$

This means that $\int_1^\infty e^{-x} dx$ is convergent. By the Comparison Test, so is $\int_1^\infty e^{-x^2} dx$.

(b) (V) Notice that

$$\frac{1+e^{-x}}{x} > \frac{1}{x}.$$

Since $\int_1^\infty \frac{1}{x} dx$ is divergent, by the Comparison Test, so is $\int_1^\infty \frac{1+e^{-x}}{x} dx$.

(c) (V) Notice that $\ln(k) > 1$ for all $k \geqslant 3$. So,

$$\frac{\ln(k)}{k} > \frac{1}{k},$$

for every $k \geqslant 3$. Since $\sum \frac{1}{k}$ is divergent, by the Comparison Test, so is $\sum \frac{\ln(k)}{k}$.

(d) (F) Since $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$ for $p \leqslant 0$, we have that $\sum \frac{1}{n^p}$ diverges.

(e) (V) Notice that we have

$$\frac{2020}{2n^2+4n+3} \leqslant \frac{2020}{2n^2} = \frac{2020}{2} \cdot \frac{1}{n^2}.$$

This means that

$$\sum \frac{2020}{2n^2+4n+3} \leqslant \frac{2020}{2} \sum \frac{1}{n^2}.$$

Since $\sum \frac{1}{n^2}$ is convergent, by the Comparison Test, so it $\sum \frac{2020}{2n^2+4n+3}$.

**Exercise 3**:

(a) By using integration by parts with $u = x$ and $dv = \sin(x)dx$, we get that

$$\int x\sin(x)dx = -x\cos(x) + \sin(x) + C.$$

(b) By using partial fractions with

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4},$$

we get that

$$\int \frac{2x^2-x+4}{x^3+4x}dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx.$$

Now, by using the substitution $u = x^2$ and using the fact that $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$, we get that

$$\int \frac{2x^2-x+4}{x^3+4x}dx = \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

(c) Using that $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ and the substitution $u = 2x$, we get that

$$\int_0^{2\pi} \sin^2(x)dx = \pi.$$

**Exercise 4**:

(a) If $f(x) = e^x$, we have that $f^{(n)}(0) = e^0 = 1$ for every $n \in \mathbb{N}$. Therefore, the Maclaurin series is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(b) If $a_n = \frac{x^n}{n!}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^n \cdot x}{(n+1)\cdot n!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1.$$

By the Ratio Test, the radius of convergence is $R = \infty$.

(c) Substituting $x$ with $x^{-2}$ in the series of $e^x$, we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!},$$

which has the same radius of convergence of $e^x$. Integrating term-by-term the series for $e^{-x^2}$, we get that

$$\int e^{-x^2} dx = C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}.$$

**Exercise 5**:

(a) See figure below.

(b) Solving $x^2 = 2x - x^2$, we get that the points of intersection are $(0,0)$ and $(1,1)$.

(c) The area of the region enclosed by the parabolas (blue region) is given by

$$\int_0^1 ((2x - x^2) - x^2) dx = \int_0^1 (2x - 2x^2) dx = \frac{1}{3}.$$