Calculus 1 (BE5B01MA1)
Final Exam 2 (2020, Feb, 3rd)

⋆ *This exam has* **6 problems** *with a total of* **100 points**.
⋆ *Grades classification*: **F** ($\leqslant$49pts), **E** (50-59), **D** (60-69), **C** (70-79), **B** (80-89), **A** (90-100).

Name: ______________________________

Exercise 1. [20pts] Consider the function $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

(a) [1pt] Determine the domain of f.

(b) [3pts] Evaluate $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -1^+} f(x)$.

(c) [3pts] Find the critical point(s) of f.

(d) [2.5pts] Find $f'(x)$ and the intervals where f is increasing and decreasing.

(e) [2.5pts] Does f have a local (or absolute) minimum or maximum?

(f) [5pts] Find $f''(x)$ and the intervals where f is concave upward and downward.

(g) [3pts] Use all the previous information to sketch the graph of f.

Exercise 2. [20pts] Let p be a real number.

(a) [2.5pts] Evaluate $\lim_{n \to \infty} \dfrac{1}{n^p}$ when $p < 0$ and when $p = 0$.

(b) [2.5pts] Conclude by using (a) that the series $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is divergent when $p \leqslant 0$.

(c) [5pts] Show that the harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent.

(d) [10pts] Show that $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Exercise 3. [15pts] Evaluate the following integrals.

(a) [5pts] $\int x^2 \ln(x) dx$.

(b) [5pts] $\int \tan^6(x) \sec^4(x) dx$.

(c) [5pts] $\int \dfrac{x^3 + x}{x - 1} dx$.

Exercise 4. [15pts] Consider the function $f(x) = \dfrac{e^x - 1 - x}{x^2}$.

(a) [1pts] Evaluate $\lim\limits_{x\to 0} f(x)$ using the l'Hospital's Rule.

(b) [2pts] Justify why we can apply the l'Hospital's Rule for f.

(c) [6pts] Find the Maclaurin series for e^x and its radius of convergence.

(d) [6pts] Evaluate $\lim\limits_{x\to 0} f(x)$ using the Maclaurin series for e^x.

Exercise 5. [20pts] Decide if the following statements are **true** or **false**.

- If it is **true**, explain why.
- If it is **false**, give an example that disproves the statement or explain why it is not true.

(a) [4pts] The interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ is $(-5, 1)$.

(b) [4pts] If f and g are differentiable, then $(f(x)g(x))' = f'(x)g'(x)$.

(c) [4pts] If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then by the Ratio Test the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(d) [4pts] For every real number x, we have that $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

(e) [4pts] If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent, then $\displaystyle\lim_{n\to\infty} a_n \neq 0$.

Exercise 6. [10pts] Give the definition of

(a) [2pts] a *continuous function* at a point a.

(b) [2pts] a *differentiable function* at a point a.

(c) [2pts] the *Taylor series* of a function f at a point a.

(d) [2pts] a *convergent sequence* $(a_n)_{n=1}^{\infty}$.

(e) [2pts] a *convergent series* $\sum_{n=1}^{\infty} a_n$.

Exercise 1:

(a) $D(f) = (-1, \infty)$.

(b) We have that $\lim_{x \to +\infty} f(x) = +\infty = \lim_{x \to -1^+} f(x)$.

(c) We have that

$$f'(x) = \frac{x(3x+4)}{2(x+1)^{3/2}}.$$

Note that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is **not** in the domain of f). So, the only critical point of f is 0.

(d) Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

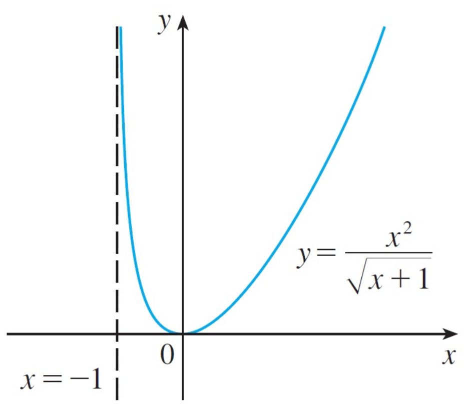
(e) Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum.

(f) We have that

$$f''(x) = \frac{3x^2 + 8x + 8}{4(x+1)^{5/2}}.$$

Notice that $f''(x) > 0$ for every $x \in (-1, \infty)$. So, f is concave upward on $(-1, \infty)$.

(g)



Exercise 2:

(a) $\lim_{n \to \infty} \dfrac{1}{n^p} = \infty$ when $p < 0$ and $\lim_{n\to\infty} \frac{1}{n^p} = 1$ when $p = 0$.

(b) If $\lim_{n \to} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) It is divergent because the partial sums $s_2, s_4, s_8, s_{16}, s_{32}, \ldots$ converges to ∞ since

$$s_{2^n} > 1 + \frac{n}{2}.$$

(d) Recall that $\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \leqslant 1$. By the Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leqslant 1$. For $p = 1$, we come back to (c) and the series diverges.

Exercise 3:

(a) Use the substitutions $u = \ln(x)$ and $dv = x^2 dx$, we get that

$$\int x^2 \ln(x) dx = \frac{1}{3} x^3 \ln(x) - \frac{1}{9} x^3 + C.$$

(b) Using that $\sec^2 x = 1 + \tan^2 x$, we have that

$$\tan^6 x \sec^4 x = \tan^6 x \sec^2 x \sec^2 x = \tan^6 x (1 + \tan^2 x) \sec^2 x.$$

So, we can use the substitution $u = \tan x$ to get

$$\int \tan^6 x \sec^4 x dx = \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.$$

(c) Since the degree of the numerator is greater than the degree of the denominator, we use the long division to get that

$$\frac{x^3 + x}{x - 1} = x^2 + x + 2 + \frac{2}{x - 1}.$$

So,

$$\int \frac{x^3 + 3}{x - 1} dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln|x - 1| + C.$$

Exercise 4:

(a) By the l'Hospital's Rule, we have that

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.$$

(b) Both $e^x - 1 - x$ and x^2 are differentiable and x^2 is different from zero around 0.

(c) If $f(x) = e^x$, we have that $f^{(n)}(0) = e^0 = 1$ for every $n \in \mathbb{N}$. Therefore, the Maclaurin series is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

If $a_n = \frac{x^n}{n!}$, then

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^n \cdot x}{(n+1) \cdot n!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \frac{|x|}{n+1} = 0 < 1.$$

By the Ratio Test, the radius of convergence is $R = \infty$.

(d) Note first that

$$e^x - 1 - x = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

So,

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4} + \dots.$$

This implies that

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2!} = \frac{1}{2}.$$

Exercise 5:

(a) (V) For every $n \in \mathbb{N}$, let

$$a_n = \frac{n(x+2)^n}{3^{n+1}}.$$

Then, we have that

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3} \frac{n}{n+1} |x+2| \longrightarrow \frac{1}{3}|x+2|.$$

By the Ratio Test, the series is convergent when $\frac{|x+2|}{3} < 1$, which is equivalent to say that $-5 < x < 1$. Now, if $x = 1$, then we have that

$$\frac{n(x+2)^n}{3^{n+1}} = \frac{n3^n}{3^{n+1}} = \frac{n}{3} \longrightarrow \infty,$$

which implies that the series is divergent. For $x = 5$, we have that

$$\frac{n(x+2)^n}{3^{n+1}} = \frac{n(-3)^n}{3^{n+1}} = \frac{(-1)^n n}{3},$$

which is not convergent and then the series is again divergent. This shows that the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ is $(-5, 1)$.

(b) (F) For instance, if $f(x) = x^2$ and $g(x) = x$, then $f'(x) = 2x$ and $g'(x) = 1$. On one hand, we have that

$$(f(x)g(x))' = (x^2 \cdot x)' = (x^3)' = 3x^2.$$

On the other hand,

$$f'(x)g'(x) = 2x \cdot 1 = 2x.$$

(c) (F) If $a_n = \frac{1}{n}$, then we have that

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{1}{n+1} \cdot n \right| \longrightarrow 1,$$

but $\sum_{n=1}^{\infty} \frac{1}{n}$ is diverges.

(d) (V) By Exercise 4(c), we have that $e^x = \sum \frac{x^n}{n!}$ is convergent for every $x \in \mathbb{R}$. So, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every $x \in \mathbb{R}$.

(e) (F) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\lim_{n\to\infty} \frac{1}{n} = 0$.