Calculus 1 (BE5B01MA1)
Final Exam 3 (2020, Feb, 7nd)

⋆ *This exam has* **6 problems** *with a total of* **100 points**.
⋆ *Grades classification*: **F** ($\leqslant$49pts), **E** (50-59), **D** (60-69), **C** (70-79), **B** (80-89), **A** (90-100).

Name: ____

Exercise 1. [20pts] Sketch the graph of a function f that satisfies all the following conditions. It must be **just one** graph and each item interpreted correctly gives you 4 points.

(i) [4pts] $f'(0) = f'(2) = f'(4) = 0$.

(ii) [4pts] $f'(x) > 0$ if $x < 0$ or $2 < x < 4$.

(iii) [4pts] $f'(x) < 0$ if $0 < x < 2$ or $x > 4$.

(iv) [4pts] $f''(x) > 0$ if $1 < x < 3$.

(v) [4pts] $f''(x) < 0$ if $x < 1$ or $x > 3$.

Exercise 2. [15pts]

(a) [6pts] Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

(b) [6pts] Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(c) [3pts] Show that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 4$.

Exercise 3. [15pts] Evaluate the following integrals.

(a) [5pts] $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} dx$.

(b) [5pts] $\displaystyle\int \sec^3(x) dx$.

(c) [5pts] $\displaystyle\int_{-1}^{2} |x| dx$.

Exercise 4. [20pts]

(a) [5pts] Find the Maclaurin series for $\sin(x)$.

(b) [5pts] Find its radius of convergence.

(c) [5pts] Evaluate $\lim_{x\to 0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$ using the L'Hospital's Rule.

(d) [5pts] Evaluate $\lim_{x\to 0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$ using the Maclaurin series for $\sin(x)$.

Exercise 5. [20pts] Decide if the following statements are **true** or **false**.

- If it is **true**, explain why.
- If it is **false**, give an example that disproves the statement or explain why it is not true.

(a) [4pts] If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) [4pts] If $0 \leqslant a_n \leqslant b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

(c) [4pts] The Ratio Test can be used to determine whether $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

(d) [4pts] If $\int_0^1 f(x)dx = 0$, then $f(x) = 0$ for $0 \leqslant x \leqslant 1$.

(e) [4pts] If $f'(c) = 0$, then f has a local maximum or minimum at c.

Exercise 6. [10pts] Give the definition of

(a) [2pts] a *continuous function* at a point a.

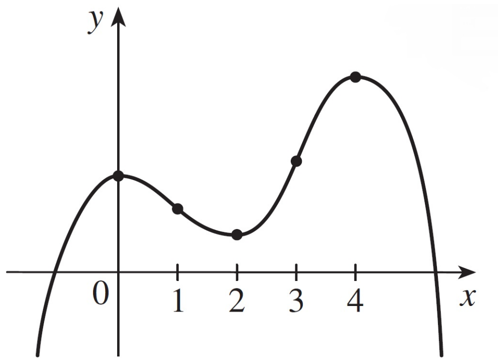
(b) [2pts] a *differentiable function* at a point a.

(c) [2pts] the *Taylor series* of a function f at a point a.

(d) [2pts] a *convergent sequence* $(a_n)_{n=1}^{\infty}$.

(e) [2pts] a *convergent series* $\sum_{n=1}^{\infty} a_n$.

Exercise 1:



Exercise 2:

(a) Notice that for every $n \in \mathbb{N}$, we have

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

So,

$$s_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1}.$$

This shows that $\lim_{n\to\infty} s_n = 1$. So, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

(b) It is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$. So,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

(c) Since both $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ are convergent, we have that

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 3 \cdot 1 + 1 = 4,$$

by using the results of items (a) and (b).

Exercise 3:

(a) By using partial fractions, we have

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2+1)^2}.$$

Solving this, we find $A = 1$, $B = -1$, $C = -1$, $D = 1$, and $E = 0$. Therefore,

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} dx = \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1} x - \frac{1}{2(x^2+1)} + C.$$

(b) We integrate by parts with $u = \sec(x)$ and $dv = \sec^2(x)dx$. Then $du = \sec x \tan x dx$ and $v = \tan x$. Then,

$$\begin{aligned}\int \sec^3 x dx &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1)dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx.\end{aligned}$$

So,

$$\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.$$

(c) We have that

$$\int_{-1}^{2} |x|dx = \int_{-1}^{0} |x|dx + \int_{0}^{2} |x|dx = \int_{-1}^{0}(-x)dx + \int_{0}^{2} xdx = \frac{5}{2}.$$

Exercise 4:

(a) Let $f(x) = \sin(x)$. Then,

$$f'(x) = \cos(x), f''(x) = -\sin(x), f'''(x) = -\cos(x), f^{(4)}(x) = \sin(x), \ldots.$$

So, $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{(4)}(0) = 0, \ldots$. Since the derivatives repeat in a cycle of four, we have can write the Maclaurin series as follows:

$$\begin{aligned}f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \frac{f^{(4)}(0)}{4!} + \ldots &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.\end{aligned}$$

(b) Let $a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^n \cdot (-1) \frac{x^{2n+1} \cdot x^2}{(2n+3) \cdot (2n+1)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}}\right| = \frac{x^2}{2n+3} \longrightarrow 0.$$

Therefore, $R = \infty$.

(c) Applying the L'Hospital's Rule five times, we have that

$$\lim_{x \to 0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5} = \lim_{x \to 0} \frac{\cos(x)}{5!} = \frac{1}{5!}.$$

(d) Notice first that

$$\sin(x) - x + \frac{1}{6}x^3 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) - x + \frac{1}{6}x^3 = \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots.$$

Then,

$$\frac{\sin(x) - x + \frac{1}{6}x^3}{x^5} = \frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \dots$$

and so $\lim_{x\to 0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5} = \frac{1}{5!}$.

Exercise 5:

(a) (F) If $a_n = \frac{1}{n}$, then $\lim_{n\to\infty} a_n = 0$ but $\sum a_n$ diverges.

(b) (F) We have that $0 \leqslant \frac{1}{n^2} \leqslant \frac{1}{n}$ for every $n \in \mathbb{N}$, $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges.

(c) (F) The Ratio Test is inconclusive since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

(d) (F) For instance, let $f(x) = x - \frac{1}{2}$. Then,

$$\int_0^1 f(x)dx = \left[\frac{x^2}{2} - \frac{1}{2}x\right]_0^1 = 0$$

and $f(x) \neq 0$ on $(0, 1)$.

(e) (F) Consider, for instance, $f(x) = x^3$. We have that $f'(0) = 0$ but f does not have neither a local maximum nor a local minimum at 0.