Calculus 1 (BE5B01MA1)
Lab 14

Recall the following Maclaurin series that we have seen in our lessons.

- $\star$ $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ with $R = 1$.
- $\star$ $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ with $R = \infty$.
- $\star$ $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$ with $R = \infty$.
- $\star$ $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$ with $R = \infty$.
- $\star$ $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ with $R = 1$.
- $\star$ $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$ with $R = 1$.
- $\star$ $(1+x)^k = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^2 + \ldots$ with $R = 1$.

Exercise 1 (Taylor series)**.** Find the Taylor series for $f(x)$ centered at the given value of a. Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.

(1) $f(x) = x^4 - 3x^2 + 1$, $a = 1$.

(2) $f(x) = x - x^3$, $a = -2$.

(3) $f(x) = \ln(x)$, $a = 2$.

(4) $f(x) = e^{2x}$, $a = 3$.

(5) $f(x) = \cos(x)$, $a = \pi$.

(6) $f(x) = 1/x$, $a = -3$.

(7) $f(x) = \sin(x)$, $a = \pi/2$.

(8) $d(x) = \sqrt{x}$, $a = 16$.

Exercise 2 (Maclaurin series)**.** Use a Maclaurin series to obtain the Maclaurin series for the given function.

(1) $f(x) = \sin(\pi x)$.

(2) $f(x) = e^x + e^{2x}$

(3) $f(x) = x\cos\left(\frac{1}{2}x^2\right)$.

(4) $f(x) = \sin^2(x)$.

(5) $f(x) = x^2\ln(1+x)$.

(6) $f(x) = \frac{x}{\sqrt{4+x^2}}$.

(7) $f(x) = \frac{x^2}{\sqrt{2+x}}$.

Exercise 3. Use series to evaluate the following limits.

$$\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2} \qquad \lim_{x\to 0}\frac{1-\cos(x)}{1+x-e^x} \qquad \lim_{x\to 0}\frac{\sin(x)-x+\frac{1}{6}x^3}{x^5}$$

Exercise 4. Evaluate the indefinite integral as an infinite series.

$$\int x\cos(x^3)dx \qquad \int \frac{e^x-1}{x}dx \qquad \int \frac{\cos(x)-1}{x}dx \qquad \int \arctan(x^2)dx.$$

Exercise 5. Find the first 3 nonzero terms in the Maclaurin series for:

(a) $e^x\sin(x)$.

(b) $\tan(x)$.

Hint: For (a), use the Maclaurin series for e^x and $\sin(x)$, then multiply these expressions, collecting like terms just as for polynomials. For (b), use that $\tan(x) = \frac{\sin(x)}{\cos(x)}$, $\sin(x)$ and $\cos(x)$ Maclaurin series, and procedure like long division, again as for polynomials.

Exercise 6 (Group discussion)**.** Consider the function $f(x) = \cos(x^2)$.

(a) Find the Maclaurin series of f and its radius of convergence.

(b) Graph f and its first 3 Taylor polynomials on the same screen.

(c) What do you notice about the relationship between these polynomials and f?