On a vector valued Bollobás theorem for compact operators

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Joint work with D. García, M. Maestre and M. Martín
Alcoy, July 15th, 2016

Definitions & Some Results

Definition - Norm Attaining Functional

We say that a linear functional $x^* \in X^*$ **attains its norm** if there exists $x_0 \in S_X$ such that $|x^*(x_0)| = \|x^*\|$. The set of all norm attaining functionals is denoted by $NA(X)$.

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James Theorem, 1957

A Banach space is reflexive if and only if every bounded linear functional attains its norm.

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Bishop-Phelps Theorem, 1961

Every element in X^* can be approximated by a norm attaining linear functional. In other words, $\overline{NA(X)} = X^*$.

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Question (Bishop-Phelps)

Is it true for bounded linear operators?

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We say that a bounded linear operator $T \in \mathcal{L}(X,Y)$ **attains its norm** if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\|$. The set of all norm attaining operators is denoted by $NA(X,Y)$.

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Lindenstrauss' counterexample, 1963

There exists a Banach space X such that

$$\overline{NA(X, X)} \neq \mathcal{L}(X, X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators.

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Open Question

$\overline{NA(X,\mathbb{R}^2)} = \mathcal{L}(X,\mathbb{R}^2)$ for all Banach X?

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In 1970, Bollobás proved a very useful theorem to study numerical radius of operators:

Bishop-Phelps-Bollobás Theorem, 1970 (Martín's version, 2014)

Let X be a Banach space and $\varepsilon \in (0, 2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2}$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = |y^*(y)| = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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Observation

It is **not expected** that there exists a Bishop-Phelps-Bollobás theorem version for bounded linear operators.

BPBp for compact operators

Because of this, Acosta, Aron, García and Maestre defined the

Bishop-Phelps-Bollobás property

A pair of Banach spaces (X, Y) has the **BPBp** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exist $S \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

BPBp for compact operators

BPBp for compact operators

A pair of Banach spaces (X, Y) has the **BPBp for compact operators** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in S_{\mathcal{K}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exist $S \in S_{\mathcal{K}(X,Y)}$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

BPBp for compact operators

Examples

BPBp for compact operators

Examples

(a) X is uniformly convex and Y is any Banach space.
[Canad. J. Math., S. K. Kim and H. J. Lee]
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(c) $X = L_1(\mu)$ and $Y = L_1(\nu)$ for arbitrary measures μ and ν.
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(d) $X = L_1(\mu)$ and $Y = L_\infty(\nu)$ for any measure μ and any localizable measure ν
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(e) $X = C(K_1)$ and $Y = C(K_2)$ in the real case
[2014, Nonlinear Anal., S. K. Kim, H. J. Lee, et al.,]

BPBp for compact operators

Examples

(f) $X = C_0(L)$ and Y uniformly convex for L a locally compact Hausdorff topological space
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(g) X is a arbitrary Banach space and Y^* is isometrically isomorphic to a $L_1(\mu)$-space
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(h) $X = L_1(\mu)$ for a arbitrary measure μ and Y having the AHSP
[2014, JMAA, Acosta, Becerra-Guerrero, García, Kim and Maestre]

BPBp for compact operators

Martín's counterexample:

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[JFA, 2014, M. Martín, Norm attaining compact operators]

There exist compact operators which cannot be approximated by norm attaining ones.

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Martín's counterexample:

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There exist compact operators which cannot be approximated by norm attaining ones.

The counterexample uses the 1973 Enflo's counterexample which shows that there exists a closed subspace of c_0 without the approximation property.

BPBp for compact operators

- In [JFA, 2014, M. Martín, Norm attaining compact operators], the author studied the conditions that X and Y must have in order to approximate compact operators by norm attaining ones.

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- In [JFA, 2014, M. Martín, Norm attaining compact operators], the author studied the conditions that X and Y must have in order to approximate compact operators by norm attaining ones.
- We did the same thing but now considering the BPBp for compact operators.

BPBp for compact operators

Technical Lemma (domain space)

Let X and Y be Banach spaces. Suppose that there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that given $\delta > 0$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ and $x_0 \in S_X$, we may find a norm one operator $P \in \mathcal{L}(X, X)$ and a norm one operator $i \in \mathcal{L}(P(X), X)$ such that

(i) $\|P^* x_j^* - x_j^*\| < \delta$, for all $j = 1, \ldots, n$

(ii) $\|i(P(x_0)) - x_0\| < \delta$,

(iii) $P \circ i = Id_{P(X)}$,

(iv) the pair $(P(X), Y)$ has the BPBp for compact operators with the function η.

Then the pair (X, Y) has the BPBp for compact operators.

BPBp for compact operators

Theorem

Let L be a locally compact Hausdorff topological space and let Y be a Banach space. If (c_0, Y) has the BPBp for compact operators, then $(C_0(L), Y)$ has the BPBp for compact operators.

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Theorem

Let L be a locally compact Hausdorff topological space and let Y be a Banach space. If (c_0, Y) has the BPBp for compact operators, then $(C_0(L), Y)$ has the BPBp for compact operators.

- When Y is **uniformly convex**, (c_0, Y) has the BPBp for compact operators.

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- When Y is **uniformly convex**, (c_0, Y) has the BPBp for compact operators.
- When Y has **property** β, (c_0, Y) has the BPBp for compact operators.

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Theorem

Let Y be a Banach space. If the pair (c_0, Y) has the BPBp, then (c_0, Y) has the BPBp for compact operators.

BPBp for compact operators

Corollary

Let L be a locally compact Hausdorff topological space and let Y be a Banach space. Suppose that there exist a set I, $\{y_i : i \in I\} \subset S_Y$, $\{y_i^* : i \in I\} \subset S_{Y^*}$, a subset $E \subset S_Y$, a mapping $F : E \longrightarrow S_{Y^*}$ and $0 \leq \rho < 1$ satisfying that

(1) $y_i^*(y_i) = 1, \forall i \in I$;

(2) $|y_i^*(y_j)| \leq \rho, \forall i, j \in I, i \neq j$;

(3) E is uniformly strongly exposed by F;

(4) $|F(e)(y_i)| \leq \rho, \forall e \in E, i \in I$;

(5) for any $y \in Y$,
$$\|y\| = \max\big\{\sup\{|y_i^*(y)| : i \in I\}, \sup\{|F(e)(y)| \;:\; e \in E\}\big\}.$$

Then, the pair $(C_0(L), Y)$ has the BPBp for compact operators.

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Corollary

Let L be a locally compact Hausdorff topological space and **let Y be a Banach space**. Suppose that there exist a set I, $\{y_i : i \in I\} \subset S_Y$, $\{y_i^* : i \in I\} \subset S_{Y^*}$, a subset $E \subset S_Y$, a mapping $F : E \longrightarrow S_{Y^*}$ and $0 \leq \rho < 1$ satisfying that

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Then, the pair $(C_0(L), Y)$ has the BPBp for compact operators.

BPBp for compact operators

Technical lemma

Let X be Banach space for which there exists a net $\{P_\alpha\}_{\alpha\in\Lambda}$ of rank-one projections on X such that

(i) $\{P_\alpha x\} \longrightarrow x$ for all $x \in X$ in norm and

(ii) $\{P_\alpha^* x^*\} \longrightarrow x^*$ for all $x^* \in X^*$ in norm.

If for a Banach space Y there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that all the pairs $(P_\alpha(X), Y)$ with $\alpha \in \Lambda$ have the BPBp for compact operators with the function η, then the pair (X, Y) has the BPBp for compact operators.

BPBp for compact operators

Theorem

Let μ be a positive measure, let X be a Banach space such that X^* has the Radon-Nikodým property and let Y be a Banach space. If $(\ell_1(X), Y)$ has the BPBp for compact operators, then the pair $(L_1(\mu, X), Y)$ has the BPBp for compact operators.

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Let μ be a positive measure and let X, Y be Banach spaces. The pair $(L_1(\mu, X), Y)$ has the BPBp for compact operators in the following cases:

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Let μ be a positive measure and let X, Y be Banach spaces. The pair $(L_1(\mu, X), Y)$ has the BPBp for compact operators in the following cases:

(a) if X and Y are finite-dimensional;

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Let μ be a positive measure and let X, Y be Banach spaces. The pair $(L_1(\mu, X), Y)$ has the BPBp for compact operators in the following cases:

(a) if X and Y are finite-dimensional;

(b) if Y is a Hilbert space and $X = c_0$ or $X = L_p(\nu)$ for any positive measure ν and $1 < p < \infty$.

BPBp for compact operators

Auxiliary result

Let Y be a Banach space. Then, the following are equivalent:

(i) the pair (ℓ_1, Y) has the BPBp for compact operators;

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(iv) for every positive measure μ, the pair $(L_1(\mu), Y)$ has the BPBp for compact operators;

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(i) the pair (ℓ_1, Y) has the BPBp for compact operators;

(ii) Y has the AHSP;

(iii) the pair (ℓ_1, Y) has the BPBp;

(iv) for every positive measure μ, the pair $(L_1(\mu), Y)$ has the BPBp for compact operators;

(v) there is a positive measure μ such that $L_1(\mu)$ is infinite-dimensional and the pair $(L_1(\mu), Y)$ has the BPBp for compact operators.

BPBp for compact operators

Technical lemma

Let X be a Banach space. Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a net of norm-one projections on X such that

(i) $\alpha \preceq \beta$ implies that $P_\alpha(X) \subset P_\beta(X)$ and

(ii) $\{P_\alpha^* x^*\} \longrightarrow x^*$ in norm for all $x^* \in X^*$.

If for a Banach space Y there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that all the pairs $(P_\alpha(X), Y)$ with $\alpha \in \Lambda$ have the BPBp for compact operators with the function η, then the pair (X, Y) has the BPBp for compact operators.

BPBp for compact operators

Theorem

Let X be a Banach space such that X^* is isometrically isomorphic to ℓ_1 and let Y be a Banach space. If the pair (c_0, Y) has the BPBp for compact operators, then (X, Y) has the BPBp for compact operators.

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- When Y is **uniformly convex**, (c_0, Y) has the BPBp for compact operators.

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BPBp for compact operators

Corollary

Let X be a Banach space such that X^* is isometrically isomorphic to ℓ_1 and **let Y be a Banach space**. Suppose that there exist a set I, $\{y_i : i \in I\} \subset S_Y$, $\{y_i^* : i \in I\} \subset S_{Y^*}$, a subset $E \subset S_Y$, a mapping $F : E \longrightarrow S_{Y^*}$ and $0 \leq \rho < 1$ satisfying that

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Then, the pair (X, Y) has the BPBp for compact operators.

BPBp for compact operators

Technical lemma (range space)

Let X and Y be Banach spaces. Suppose that there exists a net of **norm-one** projections $\{Q_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{L}(X, Y)$ such that

$$\{Q_\lambda y\} \longrightarrow y \text{ in norm for every } y \in Y.$$

If there is a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the pairs $(X, Q_\lambda(Y))$ with $\lambda \in \Lambda$ have the BPBp for compact operators with the function η, then the pair (X, Y) has the BPBp for compact operators.

BPBp for compact operators

Lemma

Let X, Y be Banach spaces and let $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a function. The following are equivalent:

(i) the pair (X, Y) has the BPBp for compact operators with the function η,

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Lemma

Let X, Y be Banach spaces and let $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a function. The following are equivalent:

(i) the pair (X, Y) has the BPBp for compact operators with the function η,

(ii) the pairs $(X, \ell_\infty^m(Y))$ with $m \in \mathbb{N}$ have the BPBp for compact operators with the function η,

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Lemma

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Lemma

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(iv) the pair $(X, \ell_\infty(Y))$ has the BPBp for compact operators with the function η.

BPBp for compact operators

Theorem

Let X, Y be Banach spaces.

(a) For $1 \leq p < \infty$, if the pair $(X, \ell_p(Y))$ has the BPBp for compact operators, then so does $(X, L_p(\mu, Y))$ for every positive measure μ such that $L_1(\mu)$ is infinite-dimensional.

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(b) If the pair (X, Y) has the BPBp for compact operators, then so does $(X, L_\infty(\mu, Y))$ for every σ-finite positive measure μ.

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Theorem

Let X, Y be Banach spaces.

(a) For $1 \leq p < \infty$, if the pair $(X, \ell_p(Y))$ has the BPBp for compact operators, then so does $(X, L_p(\mu, Y))$ for every positive measure μ such that $L_1(\mu)$ is infinite-dimensional.

(b) If the pair (X, Y) has the BPBp for compact operators, then so does $(X, L_\infty(\mu, Y))$ for every σ-finite positive measure μ.

(c) If the pair (X, Y) has the BPBp for compact operators, then so does $(X, C(K, Y))$ for every compact Hausdorff topological space K.

BPBp for compact operators

Corollary

Let X, Y be Banach spaces, let K be a compact Hausdorff topological space, let μ be a positive measure and let ν be a σ-finite positive measure.

(a) If Y has property β, then $(X, L_\infty(\mu, Y))$ and $(X, C(K, Y))$ have the BPBp for compact operators.

BPBp for compact operators

Corollary

Let X, Y be Banach spaces, let K be a compact Hausdorff topological space, let μ be a positive measure and let ν be a σ-finite positive measure.

(a) If Y has property β, then $(X, L_\infty(\mu, Y))$ and $(X, C(K, Y))$ have the BPBp for compact operators.

(b) If Y has the AHSP, then so do $L_\infty(\nu, Y)$ and $C(K, Y)$.

BPBp for compact operators

Corollary

Let X, Y be Banach spaces, let K be a compact Hausdorff topological space, let μ be a positive measure and let ν be a σ-finite positive measure.

(a) If Y has property β, then $(X, L_\infty(\mu, Y))$ and $(X, C(K, Y))$ have the BPBp for compact operators.

(b) If Y has the AHSP, then so do $L_\infty(\nu, Y)$ and $C(K, Y)$.

(c) For $1 \leq p < \infty$, if $\ell_p(Y)$ has the AHSP and $L_1(\mu)$ is infinite-dimensional, then $L_p(\mu, Y)$ has the AHSP as well.

BPBp for compact operators

Thank you very much for your attention!