Strongly subdifferentiability and the Bollobás theorem

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Strong subdifferentiability of the norm

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We say that the norm of a Banach space X is strongly subdifferentiable (**SSD**, for short) at a point $u \in S_X$ if the one-sided limit

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- Consider ϕ_n on B_X defined by

$$\phi_n(x) = \frac{1}{n}\left(\left\|u + \frac{x}{n}\right\| - 1\right) = \|nu + x\| - n.$$

- Then, the norm of X is SSD iff $\{\phi_n\}$ converges uniformly on B_X.

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Theorem (C. Franchetti and R. Payá, 1993)

The pair $(X, \mathbb{K})$ has the property $\star$ iff X is SSD.

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$$1 \geq |x_n^*(x_0)| \geq 1 - \frac{1}{n}$$

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By the Banach-Alaoglu theorem, there is a subnet of (x_n^*) such that $x_n^* \xrightarrow{w^*} x_0^*$ for some $x_0^* \in B_{X^*}$.

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- The trace class $\mathcal{C}_1$.

are non-reflexive dual spaces that satisfy the w^*-Kadec-Klee property.

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Examples of linear operators

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- (X, Y) has property $\star$ for finite-dimensional spaces X, Y.
- (ℓ_1^N, X) has property $\star$ when X is uniformly convex.
- $(c_0, L_p(\mu))$ has property $\star$ for μ positive measures and $1 \leq p < \infty$.

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$$|\hat{A}(x \otimes y)| = |A(x, y)| > 1 - \eta(\varepsilon, x \otimes y).$$

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- Recall that $\mathcal{L}(\ell_s, \ell_r) = \mathcal{L}(\ell_s, \ell_{r'}; \mathbb{K}) = (\ell_s \hat{\otimes}_\pi \ell_{r'})^*$.
- Since $\ell_s \hat{\otimes}_\pi \ell_{r'}$ is reflexive, then $B_{(\ell_s \hat{\otimes}_\pi \ell_{r'})^*}$ is w^*-sequentially compact and then $(\ell_s \hat{\otimes}_\pi \ell_{r'})^*$ satisfies the sequential-w^*-uniform-Kadec-Klee property for $1 < r < 2 < s < \infty$ (S.J. Dilworth and D. Kutzarova, 1995).

Fix $\varepsilon > 0$ and $(x, y) \in S_{\ell_p} \times S_{\ell_q}$. Consider $\eta(\varepsilon, x \otimes y) > 0$. Let $A \in \mathcal{L}(\ell_p, \ell_q; \mathbb{K})$ with $\|A\| = 1$ with

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Consider $\hat{A} \in S_{(\ell_p \hat{\otimes}_\pi \ell_q)^*}$. Then,

$$|\hat{A}(x \otimes y)| = |A(x, y)| > 1 - \eta(\varepsilon, x \otimes y).$$

Then, there is $\hat{B} \in S_{(\ell_p \hat{\otimes}_\pi \ell_q)^*}$ with

$$|B(x, y)| = |\hat{B}(x \otimes y)| = 1 \quad \text{and} \quad \|B - A\| = \|\hat{B} - \hat{A}\| < \varepsilon.$$

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- Then, $(\ell_p \hat{\otimes}_\pi \ell_q; \mathbb{K})$ has the $\star$ for $2 < p, q < \infty$.

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SSD in projective tensor products

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As a consequence...

(a). If $2 < p, q < \infty$, then $\ell_p \hat{\otimes}_\pi \ell_q$ is SSD.

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(c). If $p^{-1} + q^{-1} \geq 1$ or one of them is 1 or ∞, then $\ell_p \hat{\otimes}_\pi \ell_q$ is **not** SSD.

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(2) $\exists$ more Banach spaces X and Y such that $X \hat{\otimes}_\pi Y$ is SSD?

Thank you
for your attention