The Strong Bishop-Phelps-Bollobás property

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Definitions & Some Results

Definition - Norm Attaining Functional

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Question (Bishop-Phelps)

Is it true for operators?

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We say that a bounded linear operator $T \in \mathcal{L}(X, Y)$ **attains its norm** if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\|$. The set of all norm attaining operators is denoted by $NA(X, Y)$.

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Lindenstrauss' counterexample, 1963

There exists a Banach space X such that

$$\overline{NA(X, X)} \neq \mathcal{L}(X, X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators.

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Bishop-Phelps-Bollobás Theorem, 1970 (Martín's version, 2014)

Let X be a Banach space and $\varepsilon \in (0,2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = y^*(y) = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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Observation

It is **not** expected that there exists a Bishop-Phelps-Bollobás Theorem version for bounded linear operators.

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Definition - Bishop-Phelps-Bollobás property (BPBp)

A pair of Banach spaces (X, Y) is said to have the **BPBp** if for every $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

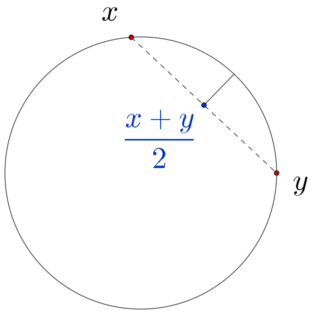
there exist $S \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

sBPBp

A Banach space X is **uniformly convex** if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$x, y \in S_X \text{ and } \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon).$$



sBPBp

In 2014, Kim and Lee proved that

Kim-Lee Theorem

A Banach space X is **uniformly convex** if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x \in B_X$ satisfy

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Is it true for operators?

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Question

Is it true for operators? **This is a hard question!**

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Definition of the sBPBp

We say that the pair of Banach spaces (X, Y) has the **strong BPBp** if given $\varepsilon \in (0,1)$ and $T \in S_{\mathcal{L}(X,Y)}$, there exists $\eta(\varepsilon, T) > 0$ such that whenever $x_0 \in S_X$ satisfies

$$\|T(x_0)\| > 1 - \eta(\varepsilon, T),$$

there exists $x_1 \in S_X$ such that

$$\|T(x_1)\| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

sBPBp

Theorem 1

Let X be a finite dimensional Banach space. Then the pair (X, Y) has the sBPBp for all Banach spaces Y.

sBPBp

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Let X be a finite dimensional Banach space. Then the pair (X, Y) has the sBPBp for all Banach spaces Y.

Theorem 2

Let X be a uniformly convex Banach space. Then the pair (X, Y) has the sBPBp *for compact operators* for all Banach spaces Y.

sBPBp

Corollary 3

If X is a uniformly convex Banach space and Y is a Banach space with the Schur's property, then the pair (X, Y) has the sBPBp. In particular, the pair (ℓ_2, ℓ_1) has the sBPBp.

sBPBp

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Corollary 4

If X is a uniformly convex Banach space and Y is a finite dimensional Banach space, then the pair (X, Y) has the sBPBp.

sBPBp

Counterexample

If X is not reflexive, then the pair (X, Y) can not have the sBPBp by the James Theorem.

Uniform sBPBp

Definition

We say that a pair of Banach space (X, Y) has the **uniform sBPBp** if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

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The Kim-Lee Theorem says that the pair $(X, \mathbb{K})$ has the uniform sBPBp if and only if X is a uniformly convex Banach space.

Uniform sBPBp

Counterexample

Consider $X = \ell_2^2(\mathbb{K})$ and $Y = \ell_\infty^2(\mathbb{K})$.

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Consider $X = \ell_2^2(\mathbb{K})$ and $Y = \ell_\infty^2(\mathbb{K})$. Suppose that there exists $\eta(\varepsilon) > 0$ with the above property. Let $T : X \to Y$ defined by

$$T(x, y) := \left(\left(1 - \frac{1}{2}\eta(\varepsilon) \right) x, y \right).$$

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So:

- $\|T\| = 1$,

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So:

- $\|T\| = 1$,
- $\|T(e_1)\|_\infty > 1 - \eta(\varepsilon)$,
- every $z \in S_X$ such that $\|T(z)\|_\infty = 1$ assumes the form $z = \lambda e_2$ for some $|\lambda| = 1$.

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But, in this case, we have $\|e_1 - z\|_2 = \sqrt{2}$.

Uniform sBPBp

All the following pairs **fail** to have the uniform sBPBp:

(1) $(\ell_2^2, \ell_\infty^2)$,

(2) (ℓ_2^2, ℓ_2^2),

(3) (ℓ_p^2, ℓ_q^2) for $1 < p \leq q < \infty$.

(4) (ℓ_2^2, ℓ_1^2),

(5) (ℓ_2^2, ℓ_q^2), for $1 \leq q < 2$.

(6) $(\ell_2^2(\mathbb{R}), \ell_q^2(\mathbb{R}))$ for $1 \leq q \leq \infty$.

(7) $(\ell_p^2(\mathbb{R}), \ell_q^2(\mathbb{R}))$ for $1 < p \leq 2$ and $1 \leq q \leq 2$.

(8) $(\ell_2^2, C[0,1])$,

(9) (Y, Y), where $\dim(Y) = 2$.

Uniform sBPBp vs sBPBp

Next, we use the negative results about the uniform sBPBp to get negative results about the sBPBp.

Uniform sBPBp vs sBPBp

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Theorem 5

If the pair (X, Y) fails the uniform sBPBp, then the pair $(\ell_2(X), \ell_\infty(Y))$ fails the sBPBp.

Uniform sBPBp vs sBPBp

In particular, the pairs $(\ell_2, \ell_\infty(Z))$ fail the sBPBp when

(a) $Z = \ell_\infty^2,\ \ \ell_2^2,\ \ \ell_1^2,\ \ C[0,1],\ \ell_q^2$ for $2 \leq q < \infty$ in both real and complex cases and

(b) $Z = \ell_q^2,\ \ell_q^2$ for $1 \leq q \leq 2$ in the real case.

Uniform sBPBp vs sBPBp

In the next theorem we give a complete characterization of the strong Bishop-Phelps-Bollobás property for the pairs (ℓ_p, ℓ_q).

Theorem 6

The following holds.

(i) The pair (ℓ_p, ℓ_q) **has** the sBPBp whenever $1 \leq q < p < \infty$.

(ii) The pair (ℓ_p, ℓ_q) **fails** the sBPBp whenever $1 < p \leq q < \infty$.

Thank you very much for your attention.