The Bishop-Phelps-Bollobás property for compact operators

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Joint work with D. García, M. Maestre and M. Martín
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Definitions & Some Results

Definition - Norm Attaining Functional

We say that a linear functional $x^* \in X^*$ **attains its norm** if there exists $x_0 \in S_X$ such that $|x^*(x_0)| = \|x^*\|$. The set of all norm attaining functionals is denoted by $NA(X)$.

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James's Theorem, 1957

If every bounded linear functional on a Banach space is norm attaining, then the space is reflexive.

Definitions & Some Results

Bishop-Phelps's Theorem, 1961

Every element in X^* can be approximated by a norm attaining linear functional. In other words, $\overline{NA(X)} = X^*$.

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Question (Bishop-Phelps)

Is it true for bounded linear operators?

Definitions & Some Results

Definition - Norm Attaining Operators

We say that a bounded linear operator $T \in \mathcal{L}(X, Y)$ **attains its norm** if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\|$. The set of all norm attaining operators is denoted by $NA(X, Y)$.

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Lindenstrauss' counterexample, 1963

There exists a Banach space X such that

$$\overline{NA(X, X)} \neq \mathcal{L}(X, X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators.

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Bishop-Phelps-Bollobás Theorem, 1970 (Martín's version, 2014)

Let X be a Banach space and $\varepsilon \in (0,2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = |y^*(y)| = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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Observation

It is **not** expected that there exists a Bishop-Phelps-Bollobás theorem version for bounded linear operators.

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Definition - Bishop-Phelps-Bollobás property (BPBp)

A pair of Banach spaces (X, Y) is said to have the **BPBp** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exist $S \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

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- (3.4) $Y = C(K)$ for K a compact Haurdorff space.

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[JFA, 2008, Acosta, Aron, García, Maestre - The BPB theorem for operators]

Definition & Some Results

4. COLLECTION OF THE CLASSIC BANACH SPACES WHICH HAVE THE BPBp FOR OPERATORS

		RANGE SPACES																				
		FD	ℓ_1^n	ℓ_p^n	ℓ_q^n	ℓ_∞^n	c_0	ℓ_1	ℓ_p	ℓ_q	ℓ_∞	$L_1(\mu)$	$L_1(\nu)$	$L_p(\mu)$	$L_p(\nu)$	$L_q(\mu)$	$L_q(\nu)$	$L_\infty(\mu)$	$L_\infty(\nu)$	$C(K)$	$C(S)$	$C_0(S)$	$C_0(L)$
DOMAIN SPACES	FD	✓	✓	✓	✓	✓																	
	ℓ_1^n	✓	✓	✓	✓	✓		✓	✓	✓		✓	✓	✓	✓	✓	✓						
	ℓ_p^n	✓	✓	✓	✓	✓		✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓				
	ℓ_q^n	✓	✓	✓	✓	✓		✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓				
	ℓ_∞^n	✓	✓	✓	✓	✓		✓ (blue)	✓	✓	✓	✓ (blue)	✓ (blue)	✓	✓	✓	✓	✓	✓				
	c_0			✓	✓				✓	✓				✓	✓	✓	✓					✓	✓
	ℓ_1	✓	✓	✓	✓			✓	✓	✓		✓	✓	✓	✓	✓	✓			✓	✓		
	ℓ_p		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
	ℓ_q		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
	ℓ_∞		✓ (blue)	✓	✓	✓		✓ (blue)	✓	✓	✓	✓ (blue)	✓ (blue)	✓	✓	✓	✓						
	$L_1(\mu)$		✓	✓	✓			✓	✓	✓		✓	✓	✓	✓								
	$L_1(\nu)$		✓	✓	✓			✓	✓	✓		✓	✓	✓	✓								
	$L_p(\mu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
	$L_p(\nu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
	$L_q(\mu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
	$L_q(\nu)$		✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓				
	$L_\infty(\mu)$		✓ (blue)	✓	✓	✓		✓ (blue)	✓	✓	✓	✓ (blue)	✓ (blue)	✓	✓	✓	✓	✓	✓				
	$L_\infty(\nu)$		✓ (blue)	✓	✓	✓		✓ (blue)	✓	✓	✓	✓ (blue)	✓ (blue)	✓	✓	✓	✓	✓	✓				
	$C(K)$			✓	✓				✓	✓				✓	✓	✓	✓			✓ (red)	✓ (red)		
	$C(S)$			✓	✓				✓	✓				✓	✓	✓	✓			✓ (red)	✓ (red)		
	$C_0(S)$		✓	✓	✓			✓	✓	✓		✓	✓	✓	✓	✓	✓					✓ (red)	✓ (red)
	$C_0(L)$		✓	✓	✓			✓	✓	✓		✓	✓	✓	✓	✓	✓					✓ (red)	✓ (red)

Cuadro 1: FD = Finite-dimensional, RED = real case and BLUE = complex case

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Definition (Compact operator)

We say that the linear operator T from X into Y is **compact** if $\overline{T(B_X)}$ is compact in Y.

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Definition (Compact operator)

We say that the linear operator T from X into Y is **compact** if $\overline{T(B_X)}$ is compact in Y.

We denote by $\mathcal{K}(X,Y)$ the set of all linear compact operators.

BPBp for compact operators

Definition (BPBp for compact operator)

A pair of Banach spaces (X, Y) has the **BPBp for compact operators** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $\boxed{T \in S_{\mathcal{K}(X,Y)}}$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exist $\boxed{S \in S_{\mathcal{K}(X,Y)}}$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

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In the proof of Theorem 2.2 of [JFA, 2008, AAGM] which says that (X, Y) has the BPBp whenever Y has the property β:

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Let $\varepsilon \in (0, 1)$, a norm one **compact operator** T and a norm one vector x_0 satisfying

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The bounded linear operator $S : X \longrightarrow Y$ defined by

$$S(x) := T(x) + [(1 + \eta)z_0^*(x) - T^*(y_{\alpha_0}^*)]y_{\alpha_0},$$

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The bounded linear operator $S : X \longrightarrow Y$ defined by

$$S(x) := T(x) + [(1+\eta)z_0^*(x) - T^*(y_{\alpha_0}^*)]y_{\alpha_0},$$

is **compact** and satisfies

$$\|S(z_0)\| = \|S\| = 1, \quad \|z_0 - x_0\| < \varepsilon \text{ and } \|S - T\| < \varepsilon.$$

for some $z_0 \in S_X$.

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More examples (X, Y) has the BPBp for compact operators if:

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More examples (X, Y) has the BPBp for compact operators if:

- X is uniformly convex.
  [2014, Canad. J. Math., S. K. Kim and H. J. Lee, Uniform convexity and the Bishop-Phelps-Bollobás property]

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- $X = L_1(\mu)$ and $Y = L_1(\nu)$ or $X = L_1(\mu)$ and $Y = L_\infty(\nu)$
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- $X = C(K_1)$ and $Y = C(K_2)$ or $X = C(K)$ and Y is uniformly convex
  [2014, Nonlinear Anal., S. K. Kim, H. J. Lee, et al., The Bishop-Phelps-Bollobás property for operators between spaces of continuous functions]

BPBp for compact operators

Martín's counterexample:

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[JFA, 2014, M. Martín, Norm attaining compact operators]

There exist compact operators which cannot be approximated by norm attaining ones.

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Martín's counterexample:

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There exist compact operators which cannot be approximated by norm attaining ones.

The counterexample uses the 1973 Enflo's counterexample which shows that there exists a closed subspace of c_0 without the approximation property.

BPBp for compact operators

- In [JFA, 2014, M. Martín, Norm attaining compact operators], the author studied the conditions that X and Y must have in order to approximate compact operators by norm attaining ones.

BPBp for compact operators

- In [JFA, 2014, M. Martín, Norm attaining compact operators], the author studied the conditions that X and Y must have in order to approximate compact operators by norm attaining ones.
- We did the same thing but now considering the BPBp for compact operators.

Positive Results in the Domain Spaces

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Technical observation: BPBp works also on the ball

A pair of Banach spaces (X, Y) has the Bishop-Phelps-Bollobás property for operators if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $\boxed{T \in B_{\mathcal{L}(X,Y)}}$ and $\boxed{x_0 \in B_X}$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there are $S \in S_{\mathcal{L}(X,Y)}$ and $x_1 \in S_X$ such that

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Positive Results in the Domain Spaces

Theorem 1

Let X and Y be Banach spaces. Suppose that there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that given $\delta > 0$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ and $x_0 \in S_X$,

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- the pair $(P(X), Y)$ has the BPBp for compact operators with the function $\eta(\varepsilon)$.

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- the pair $(P(X), Y)$ has the BPBp for compact operators with the function $\eta(\varepsilon)$.

Then the pair (X, Y) has the BPBp for compact operators.

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Proposition 2

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More specifically,

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Proposition 2 - Extension of Proposition 3.2

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(2) $\|i(P(f_0)) - f_0\| < \delta$,

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(4) $P(C_0(L)) \equiv \ell_\infty^m$ for some $m \in \mathbb{N}$.

Positive Results in the Domain Spaces

Observation

Let Y be a Banach space. The following is equivalent.

- The pair (c_0, Y) has the BPBp for compact operators.
- there is a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the pair (ℓ_∞^n, Y) with $n \in \mathbb{N}$ have the BPBp with the function η.

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Corollary 3 (Theorem 1 + Proposition 2 + Observation)

Let Y be a Banach space. If the pair (c_0, Y) has the BPBp for compact operators, then the pair $(C_0(L), Y)$ has the BPBp for compact operators.

Positive Results in the Domain Spaces

Question

Let (Ω, Σ, μ) be a measure space with μ a σ-finite measure. Is it true that the space $L_1(\mu)$ satisfies the hypothesis of Theorem 1 as well?

Positive Results in the Domain Spaces

Theorem 4

Let (Ω, Σ, μ) be a measure space with μ a σ-finite measure. Given $\varepsilon > 0$, $\{g_1, \ldots, g_n\} \subset L_\infty(\mu)$ and $\{f_1, \ldots, f_m\} \subset L_1(\mu)$, there exists a finite rank projection $P : L_1(\mu) \longrightarrow L_1(\mu)$ with $\|P\| = 1$ such that

$$\|g_i - P^* g_i\|_\infty < \varepsilon \quad \text{and} \quad \|f_j - P f_j\|_1 < \varepsilon,$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

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If the pair $(P_\alpha(X), Y)$ has the BPBp for compact operators, then the pair (X, Y) has the BPBp for compact operators.

Positive Results in the Domain Spaces

Corollary 6 (Theorem 4 + Theorem 5)

Let Y be a Banach space. If the pair (c_0, Y) has the BPBp for compact operators, then the pair $(L_1(\mu), Y)$ has the BPBp for compact operators.

Positive Results in the Domain Spaces

Corollary 6 (Theorem 4 + Theorem 5)

Let Y be a Banach space. If the pair (c_0, Y) has the BPBp for compact operators, then the pair $(L_1(\mu), Y)$ has the BPBp for compact operators.

Corollary 7 (Theorem 5)

Let $(P_\alpha)_\alpha \subset \mathcal{L}(X)$ be such that

- $P_\alpha^2 = P_\alpha$ and $\|P_\alpha\| = 1$ for all $\alpha \in \Lambda$ and

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Let $(P_\alpha)_\alpha \subset \mathcal{L}(X)$ be such that

- $P_\alpha^2 = P_\alpha$ and $\|P_\alpha\| = 1$ for all $\alpha \in \Lambda$ and
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Positive Results in the Domain Spaces

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If the pair $(P_\alpha(X), Y)$ has the BPBp for compact operators, then the pair (X, Y) has the BPBp for compact operators.

Positive Results in the Domain Spaces

ℓ_1-predual spaces

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(c) $(P_n^* x^*) \longrightarrow x^*$ in norm for all $x^* \in X^*$ and

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(c) $(P_n^* x^*) \longrightarrow x^*$ in norm for all $x^* \in X^*$ and

(d) $(P_n x) \longrightarrow x$ in norm for all $x \in X$.

[2002, Israel J. Math., I. Gasparis, On contractively complemented subspaces of separable L_1-preduals]

Positive Results in the Domain Spaces

Corollary 8 (Corollary 7 + ℓ_1-lemma)

Let X be a Banach space such that X^* is isometrically isomorphic to ℓ_1 and Y be any Banach space. If the pair (c_0, Y) has the BPBp for compact operators, then (X, Y) has the BPBp for compact operators.

Positive Results in the Range Spaces

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Theorem 9

Let X and Y be Banach spaces. Suppose that there exists $\{Q_\lambda\}_{\lambda \in \Lambda} \subset Y$ such that for all $\lambda \in \Lambda$,

Positive Results in the Range Spaces

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Let X and Y be Banach spaces. Suppose that there exists $\{Q_\lambda\}_{\lambda \in \Lambda} \subset Y$ such that for all $\lambda \in \Lambda$,

- $\|Q_\lambda\| = 1$,

Positive Results in the Range Spaces

Theorem 9

Let X and Y be Banach spaces. Suppose that there exists $\{Q_\lambda\}_{\lambda \in \Lambda} \subset Y$ such that for all $\lambda \in \Lambda$,

- $\|Q_\lambda\| = 1$,
- $Q_\lambda^2 = Q_\lambda$ and

Positive Results in the Range Spaces

Theorem 9

Let X and Y be Banach spaces. Suppose that there exists $\{Q_\lambda\}_{\lambda \in \Lambda} \subset Y$ such that for all $\lambda \in \Lambda$,

- $\|Q_\lambda\| = 1$,
- $Q_\lambda^2 = Q_\lambda$ and
- $Q_\lambda y \longrightarrow y$ in norm.

Positive Results in the Range Spaces

Theorem 9

Let X and Y be Banach spaces. Suppose that there exists $\{Q_\lambda\}_{\lambda \in \Lambda} \subset Y$ such that for all $\lambda \in \Lambda$,

- $\|Q_\lambda\| = 1$,
- $Q_\lambda^2 = Q_\lambda$ and
- $Q_\lambda y \longrightarrow y$ in norm.

If the pair $(X, Q_\lambda(Y))$ has the BPBp with a common function $\eta(\varepsilon) > 0$, then the pair (X, Y) has the BPBp for compact operators.

Positive Results in the Range Spaces

Corollary 10 (by Theorem 9)

Let X be any Banach space and Y^* isometrically isomorphic to ℓ_1. If the pair (X, ℓ_∞^n) has the BPBp with a common function, then the pair (X, Y) has the BPBp for compact operators.

This result is already know. It's the Theorem 4.2 of [2014, Nonlinear Anal., Y. S. Choi, S. K. Kim, H. J. Lee and et al., The Bishop-Phelps-Bollobás property for operators between spaces of continuous functions]

Positive Results in the Range Spaces

Theorem 11

Suppose that Y has the approximation property and that all finite-dimensional subspaces of Y have the property β with a common constant ρ. Then for every X the pair (X, Y) has the BPBp for compact operators.

Questions

(1) There are more known pairs (X, Y) satisfying the BPBp for compact operators?

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(2) There are more hypothesis that we can put in the range space to get more positive results about the BPBp for compact operators?

(3) The compactness of the operator T helped us to get positive results about the **strong BPBp**. With the all results that we mentioned in this talk, is reasonable to ask if the compactness could help in somehow in our new property until now so-called the **uniform sBPBp-p**.

Positive Results in the Range Spaces

Thank you very much for your attention!