Sobre el Teorema de Bishop-Phelps-Bollobás

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Definitions & Some Results

Definition - Norm Attaining Funcional

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Answer

Lindenstrauss gave an example of a Banach space X such that

$$\overline{NA(X,X)} \neq \mathcal{L}(X,X),$$

showing that the Bishop-Phelps result **does not** hold for operators.

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Let X be a Banach space and $\varepsilon \in (0,1)$. Given $x \in S_X$ and $x^* \in S_{X^*}$ with

$$|1 - x^*(x)| < \frac{\varepsilon^2}{4},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

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Definition - Numerical Radius

Let $T \in \mathcal{L}(X, X)$. We define the **numerical radius** of T by

$$v(T) = \sup \{|x^*(T(x))| : (x, x^*) \in \Pi(X)\}.$$

Observations & Examples

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  - The operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (-y, x)$ is such that $T \neq 0$ and $v(T) = 0$.
- Note that $v(T) \leq \|T\|$ for all $T \in \mathcal{L}(X, X)$.

Observations & Examples

Definition - Numerical Radius Attaining

We say that a operator T **attains its numerical radius** if there is $(x_0, x_0^*) \in \Pi(X)$ such that $|x^*(T(x_0))| = v(T)$.

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Is it true that the set $R(X)$ of numerical radius attaining operators is dense in $\mathcal{L}(X, X)$? **NO!**

Answer

In 1992, Payá showed that there exists a Banach space X such that the numerical radius attaining operators on X are **not dense**. Actually, X is an old example used by J. Lindenstrauss to solve the analogous problem for norm attaining operators.

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C. Cardassi 1985

$R(X)$ is dense in $\mathcal{L}(X, X)$ if $X = c_0$, $X = \ell_1$, $X = L_1(\mu)$ if μ is a finite and positive Borel measure in a compact Hausdorff topological space and for the real spaces $X = C(K)$.

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Definition - Numerical Radius for N-linear Mappings

We define the **numerical radius** of an N-linear mappings $A \in \mathcal{L}(^N X, X)$ by

$$v(A) = \sup \{|x^*(A(x_1, \ldots, x_N))| : (x^*, x_1, \ldots, x_N) \in \Pi_N(X)\},$$

where

$$\Pi_N(X) = \{(x_1, \ldots, x_N, x^*) \in S_X \times \ldots \times S_X \times S_{X^*} : x^*(x_j) = 1\}.$$

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(S. Dantas, D. García, M. Maestre, 2013)

For each $N \in \mathbb{N}$, we have that for every $A \in \mathcal{L}(^N L_1(\mu), L_1(\mu))$, $v(A) = \|A\|$.

Some Results

Question

What if we mix it up all of these concepts?

BPBp-nu for N-linear Mappings

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A Banach space X has the **BPBp-nu for $N-$linear mappings** if for every $0 < \varepsilon < 1$ there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(^N X, X)$ with $v(T) = 1$ and $x_1, \ldots, x_N \in S_X$, $x^* \in S_{X^*}$ with $x^*(x_1) = \ldots = x^*(x_N) = 1$ satisfying

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Some Recent Results

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The Banach spaces $L_1(\mu)$ (infinite dimensional) and ℓ_1 **do not have** the BPBp-nu for bilinear mappings.

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If $X = [\bigoplus_{k=1}^{\infty} X_k]_{c_0}$ or $X = [\bigoplus_{k=1}^{\infty} X_k]_{\ell_1}$ has the BPBp-nu for N-linear mappings, then each Banach space X_i has the BPBp-nu for N-linear mappings.

Thank you for your attention.