Local Bishop-Phelps-Bollobás properties

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Notation

X, Y and Z are real or complex Banach spaces.

- $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$,
- B_X is the closed unit ball of X,
- S_X is the unit sphere of X,
- $\mathcal{L}(X, Y)$ continuous linear operators from X into Y,
- $\text{NA}(X, Y)$ norm attaining operators from X into Y,
- $X^* = \mathcal{L}(X, \mathbb{K})$ topological dual of X.
- $\text{NA}(X) = \text{NA}(X, \mathbb{K})$ norm attaining on X functionals.

Motivation & History background

Bishop-Phelps theorem (1961)

Every element in X^* can be approximated by a norm attaining linear functional. In other words, $\overline{\text{NA}(X)} = X^*$.

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Is it true for bounded linear operators?

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Question (Bishop-Phelps)

Is it true for bounded linear operators?

(1963, Lindenstrauss) Counterexample

There exists a strictly convex Banach space $\mathcal{Z}$ such that the set $\text{NA}(c_0, \mathcal{Z})$ is **not** dense in $\mathcal{L}(c_0, \mathcal{Z})$.

Motivation & History background

Bishop-Phelps-Bollobás theorem (Bollobás, 1970)

Let $\varepsilon \in (0,2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$\operatorname{Re} x^*(x) > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = |y^*(y)| = 1, \quad \|y - x\| < \varepsilon, \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

(2014, M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido)

Motivation & History background

(2008, M. Acosta, R. Aron, D. García, M. Maestre)

Bishop-Phelps-Bollobás property (BPBp)

A pair of Banach spaces (X, Y) is said to have the **BPBp** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exist $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon, \quad \text{and} \quad \|T - S\| < \varepsilon.$$

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- $Y = L_1(\mu)$ with μ a finite measure.
- Y is uniformly convex.
- $Y = C(K)$ for K a compact Haurdorff space.

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- $(C(K), L_1(\mu))$ has the **BPBp**.
  (2016, M. Acosta)

The Bishop-Phelps-Bollobás point property

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(2016, D., S.K. Kim, H.J. Lee)
(2018, Y.S. Choi, D., M. Jung)
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Bishop-Phelps-Bollobás **point** property (**BPBpp**)

A pair of Banach spaces (X, Y) is said to have the **BPBpp** if for every $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exists $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ such that

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- $(X, \mathbb{K})$ has the **BPBpp** if and only if X is uniformly smooth.
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- there are uniformly smooth Banach spaces X such that the pair (X, Y) **fails** the **BPBpp** for some Y.

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- (H, Y) has the **BPBpp** for all Hilbert spaces H and any Y.

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- (H, Y) has the **BPBpp** for all Hilbert spaces H and any Y.
- $L_p(\mu)$ is **not** a **BPBpp** domain space for $p > 2$. That is, there exists Y_0 such that $(L_p(\mu), Y_0)$ **fails** the **BPBpp**.

The Bishop-Phelps-Bollobás operator property

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(2017, D.)

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Bishop-Phelps-Bollobás **operator** property (**BPBop**)

A pair of Banach spaces (X, Y) is said to have the **BPBop** if for every $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

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The Bishop-Phelps-Bollobás operator property

- $(X, \mathbb{K})$ has the **BPBop** if and only X is uniformly convex.
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- $(X, \mathbb{K})$ has the **BPBop** if and only X is uniformly convex. (2014, S.K. Kim, H.J. Lee)
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- $(X, \mathbb{K})$ has the **BPBop** if and only X is uniformly convex. (2014, S.K. Kim, H.J. Lee)
- $(X, \mathbb{K})$ has the **BPBop** if and only if $(X^*, \mathbb{K})$ has the **BPBpp**.
- If $\dim(X), \dim(Y) \geq 2$, then (X, Y) **fails** the **BPBop**. (2018, D., V. Kadets, S.K. Kim, H.J. Lee, M. Martín)

The Bishop-Phelps-Bollobás operator property

(2017, D.)
(2018, J. Talponen)
(2018, D. Sain)

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"Local **BPBop**"

A pair of Banach spaces (X, Y) is said to have the **local BPBop** if for every $\varepsilon \in (0, 1)$ and $T \in S_{\mathcal{L}(X,Y)}$, there exists $\eta(\varepsilon, T) > 0$ such that if $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon, T),$$

there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

The Bishop-Phelps-Bollobás operator property

- If X is finite dimensional, then (X, Y) has the local **BPBop** for every Banach space Y.
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- If X is finite dimensional, then (X, Y) has the local **BPBop** for every Banach space Y.
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- If X is a reflexive space with the Kadec-Klee property, then (X, Y) has the local **BPBop** for compact operators, for every Banach space Y. (2018, D. Sain)

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- If X is a reflexive space with the Kadec-Klee property, then (X, Y) has the local **BPBop** for compact operators, for every Banach space Y. (2018, D. Sain)

- It was characterized in the setting with strictly convex domain spaces and compact operators.
  (2018, J. Talponen)

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

(a) A pair (X, Y) has the $\mathbf{L}_{p,p}$ if given $\varepsilon > 0$ and $x \in S_X$, there is $\eta(\varepsilon, x) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, x),$$

there is $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ such that

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(b) A pair (X, Y) has the $\mathbf{L}_{o,o}$ if given $\varepsilon > 0$ and $T \in S_{\mathcal{L}(X,Y)}$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies

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Remark: If X is reflexive, then $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ if and only if $(X^*, \mathbb{K})$ has the $\mathbf{L}_{o,o}$.

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

To recall...

- If the one-side limit $\lim_{t\to 0^+} \frac{\|x+th\|-\|x\|}{t}$ exists uniformly for $h \in B_X$, we say that the norm of X is **SSD** at x.

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To recall...

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- When this happens for all $x \in S_X$, we say that the norm of X is **SSD**.
- When the $\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$ exists, then we say that the norm of X is Gâteaux differentiable at x and,
- if this last limit exists uniformly for all $h \in B_X$, then the norm of X is said to be Fréchet differentiable at x.

Then, X is Fréchet differentiable at x if and only if it is Gâteaux differentiable and **SSD** at x.

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

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(a) $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ if and only if the norm of X is **SSD**.

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(a) $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ if and only if the norm of X is **SSD**.

(b) $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ if and only if X is reflexive and the norm of X^* is **SSD**.

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Theorem

Suppose that the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$. The norm of X is Gâteaux differentiable if and only if it is Fréchet differentiable.

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

Definitions

(a) X is **LUR** if for all $x, x_n \in S_X$ satisfying $\lim_n \|x_n + x\| = 2$, we have that $\lim_n \|x_n - x\| = 0$.

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(b) X is *w*-**LUR** if $\lim_{n\to\infty} \|x_n + x_0\| = 2$ with $x_n, x_0 \in S_X$ implies $\lim_{n\to\infty} x_0^*(x_n) = 1$ whenever $x_0^* \in S_{X^*}$ and $x_0^*(x_0) = 1$.

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(c) X is **MLUR** if whenever $(x_n), (y_n) \subset S_X$ are norm-one sequences in S_X with $\frac{1}{2}(x_n + y_n)$ converging to some $x_0 \in S_X$, we have that $\|x_n - y_n\| \longrightarrow 0$.

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LUR $\Rightarrow$ *w*-LUR $\Rightarrow$ strict convexity
LUR $\Rightarrow$ MLUR $\Rightarrow$ strict convexity

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

Theorem

Suppose that the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$.

(a) X is strictly convex if and only if X is MLUR.

(b) X is strictly convex if and only if X^* is Fréchet differentiable. (2018, J. Talponen)

(c) X is w-LUR if and only it is LUR.

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

- X has the **Kadec-Klee property** if weak and norm topologies coincide on S_X.
- X^* has the w^*-**Kadec-Klee property** if the weak* and norm topologies coincide in S_{X^*}.

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Proposition

If X^* has the w^*-Kadec-Klee property, then the norm of X is **SSD** or, equivalently, the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$.

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Proposition

If X^* has the w^*-Kadec-Klee property, then the norm of X is **SSD** or, equivalently, the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$.

Corollary

If X is a reflexive space which satisfies the Kadec-Klee property, then the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$.
(2018, D. Sain)

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

Examples

If X is

- c_0,
- the predual of Lorentz sequence space $d_*(w,1)$,
- the space VMO (which is the predual of the Hardy space H^1),
- finite dimensional spaces ℓ_1^N, ℓ_∞^N when $N \geq 2$,

then $(X, \mathbb{K})$ **satisfies** the $\mathbf{L}_{p,p}$ but it **fails** the **BPBpp**.

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then $(X, \mathbb{K})$ **satisfies** the $\mathbf{L}_{p,p}$ but it **fails** the **BPBpp**.

Proof

Indeed, it is known that the Hardy space H^1 of analityc functions on the ball and the Lorentz spaces $L_{p,1}(\mu)$ are **non-reflexive** dual spaces that have the w^*-Kadec-Klee property.

The vector-value case

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

Proposition

- If the pair (X, Y) has the $\mathbf{L}_{p,p}$, then $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$.

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

Proposition

- If the pair (X, Y) has the $\mathbf{L}_{p,p}$, then $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$.
- If Y has property β and $(X, \mathbb{K})$ has $\mathbf{L}_{p,p}$, then so does (X, Y).

The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

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Remark: (ℓ_1, Y) **fails** the $\mathbf{L}_{p,p}$ for all Banach spaces Y, since the norm of ℓ_1 is **SSD** only at the points in the unit sphere which are sequences with finitely many nonzero terms.

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Question: We do not know whether the pair (ℓ_p^2, ℓ_q) has the $\mathbf{L}_{p,p}$ for $1 < p, q < \infty$.

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(a) A pair (X, Y) has the $\mathbf{L}_p$ if given $\varepsilon > 0$ and $x \in S_X$, there is $\eta(\varepsilon, x) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, x),$$

there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon, \quad \text{and} \quad \|S - T\| < \varepsilon.$$

The $\mathbf{L}_p$ and the $\mathbf{L}_o$

(b) A pair (X, Y) has the $\mathbf{L}_o$ if given $\varepsilon > 0$ and $T \in S_{\mathcal{L}(X,Y)}$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies

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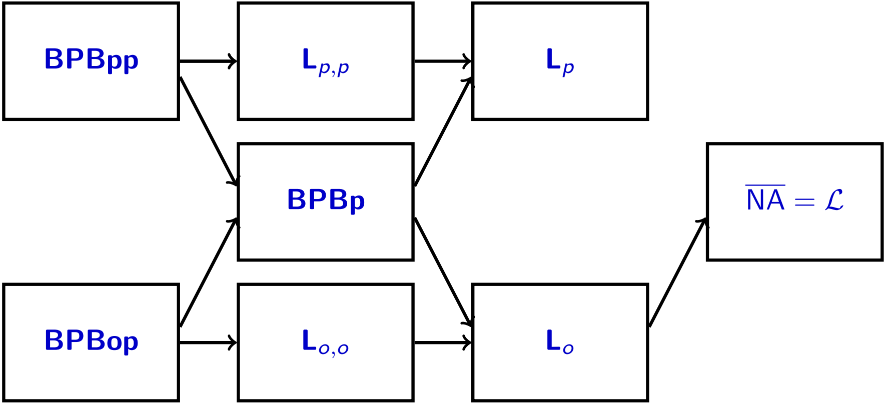
Relations between the properties



Thank you
for your attention