Norm-attaining nuclear operators

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Research supported by

Ministry of Education, Youth, and Sports of the Czech Republic

Estonian Research Council

November, 2020
Zoominar, Argentina

Joint work with
M. Jung, Ó. Roldán, and A. Rueda Zoca

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- Further Research

Motivation and Historical background

Definition

A functional $x^* \in X^*$ **attains the norm** if there is $x_0 \in S_X$ such that

$$|x^*(x_0)| = \|x^*\| = \sup_{x \in S_X} |x^*(x)|.$$

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How many functionals on X attain the norm?

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James Theorem

A Banach space X is reflexive if and only if every functional in X^* attains the norm.

Question

When is the set $\mathrm{NA}(X)$ dense in X^*?

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Bishop-Phelps Theorem (1961)

For every Banach space, $\overline{\mathrm{NA}(X)} = X^*$.

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Is it true for bounded linear operators?

Definition

$T \in \mathcal{L}(X, Y)$ **attains the norm** if there is $x_0 \in S_X$ such that

$$\|T(x_0)\| = \|T\| = \sup_{x \in S_X} \|T(x)\|.$$

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Bishop-Phelps' question

$\overline{\text{NA}(X, Y)} = \mathcal{L}(X, Y)$ for every X, Y?

Lindenstrauss counterexample (1963)

There is a Banach space X such that

$$\overline{\text{NA}(X,X)} \neq \mathcal{L}(X,X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators in general.

After this...

- Norm-attaining operators
  - J. Bourgain
  - R.E. Huff
  - W.T. Gowers
  - J. Johnson
  - W. Schachermayer
  - J.J. Uhl
  - J. Wolfe
  - V. Zizler
- Norm-attaining bilinear mappings
  - M. Acosta
  - R. Aron
  - F.J. Aguirre
  - Y.S. Choi
  - V. Lomonosov
  - R. Payá

After this...

- Norm-attaining homogeneous polynomials
    - D. Carando
    - D. García
    - S. Lassalle
    - M. Maestre
    - M. Mazzitelli
    - J.T. Rodríguez

More recently...

- B. Cascales
- R. Chiclana
- L.C. García-Lirola
- A. Guirao
- V. Kadets
- S.K. Kim
- M. Martín
- J. Merí
- V. Montesinos
- H.J. Lee
- G. López-Pérez
- D. Werner

Question (J. Diestel, J. Uhl, J. Johnson, J. Wolfe, $\approx$ 1970)

Can compact operators be approximated by norm-attaining ones?

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There exist compact operators between Banach spaces which **cannot** be approximated by norm-attaining operators.

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Answer (M. Martín, JFA, 2014)

There exist compact operators between Banach spaces which **cannot** be approximated by norm-attaining operators.

Main problem

Can finite-rank operators be approximated by norm-attaining ones?

Nuclear operators and Tensor Products

Projective tensor products

Given two Banach spaces X and Y, we denote by $X \widehat{\otimes}_\pi Y$ the projective tensor product of X and Y, which is defined as the completion of the normed space $X \otimes Y$ endowed with the norm

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$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

where the infimum is taken over all representation of z of the form $z = \sum_{i=1}^{n} x_i \otimes y_i$.

Projective tensor products

- $(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$

under the action

$$G\left(\sum_{n=1}^{\infty} x_n \otimes y_n\right) = \sum_{n=1}^{\infty} G(x_n)(y_n)$$

for $G : X \to Y^*$ as a linear functional on $X\widehat{\otimes}_\pi Y$.

Projective tensor products x Nuclear operators

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Projective tensor products x Nuclear operators

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- There is a canonical operator $J : X^* \widehat{\otimes}_\pi Y \to \mathcal{L}(X, Y)$ with $\|J\| = 1$ such that

$$u = \sum_{n=1}^{\infty} x_n^* \otimes y_n \mapsto L_u,$$

where

$$L_u(x) := \sum_{n=1}^{\infty} x_n^*(x) y_n \quad (x \in X).$$

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The operators that arise in this way are called **nuclear operators**.

Nuclear operators

We denote by $\mathcal{N}(X, Y)$ the set of all nuclear operators endowed with the norm:

$$\|T\|_N := \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n \right\},$$

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Observations

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Observations

(a) Every nuclear operator is compact.

(b) The best we can say in general is that

$$\mathcal{N}(X, Y) = X^* \widehat{\otimes}_\pi Y \big/ \ker J.$$

Norm-attainment Concepts

Norm-attaining definitions

(a) $z \in X \widehat{\otimes}_\pi Y$ **attains its projective norm** if there is a bounded sequence $(x_n, y_n) \subseteq X \times Y$ with $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$ such that $z = \sum_{n=1}^{\infty} x_n \otimes y_n$ and that $\|z\|_\pi = \sum_{n=1}^{\infty} \|x_n\| \|y_n\|$.

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(b) $T \in \mathcal{N}(X, Y)$ **attains its nuclear norm** if there is a bounded sequence $(x_n^*, y_n) \subseteq X^* \times Y$ with $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$ such that $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ and that $\|T\|_N = \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\|$.

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(c) $\mathrm{NA}_\pi(X, Y) = \{z \in X \widehat{\otimes}_\pi Y : z \text{ attains its projective norm}\}$.

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(c) $\mathrm{NA}_{\pi}(X, Y) = \{z \in X \widehat{\otimes}_{\pi} Y : z \text{ attains its projective norm}\}$.

(d) $\mathrm{NA}_{\mathcal{N}}(X, Y) = \{T \in \mathcal{N}(X, Y) : T \text{ attains its nuclear norm}\}$.

Nuclear operators and Tensors Which attain their norms

Theorem

Let X, Y be Banach spaces. Let $z \in X \widehat{\otimes}_\pi Y$ with

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$.

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(1) $z \in \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$.

(2) $\exists G \in S_{\mathcal{L}(X,Y^*)}$ such that $G(x_n)(y_n) = 1, \forall n$.

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(3) $\forall G \in S_{\mathcal{L}(X,Y^*)}$, $G(z) = \|z\|_\pi$ satisfies $G(x_n)(y_n) = 1, \forall n$.

Theorem

Let X, Y be Banach spaces. Let $T \in \mathcal{N}(X, Y)$ with

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where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$. TFAE:

(1) $T \in \text{NA}_{\mathcal{N}}(X, Y)$.

(2) $\exists G \in (\ker J)^{\perp}$ with $\|G\| = 1$ such that $G(x_n^*)(y_n) = 1, \forall n$.

(3) $\forall G \in (\ker J)^{\perp}, \|G\| = 1, G(T) = \|T\|_N \Longrightarrow G(x_n^*)(y_n) = 1, \forall n$.

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Proposition

Let H be a complex Hilbert space. Then,

(a) every nuclear operator $T \in \mathcal{N}(H, H)$ attains its nuclear norm.

(b) every tensor in $H \widehat{\otimes}_\pi H$ attains its projective norm.

It is natural to ask whether or not the equalities

$$\mathrm{NA}_{\mathcal{N}}(X, Y) = \mathcal{N}(X, Y) \quad \text{or} \quad \mathrm{NA}_{\pi}(X \widehat{\otimes}_{\pi} Y) = X \widehat{\otimes}_{\pi} Y$$

hold for every Banach spaces X and Y.

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Proposition

Let X, Y be Banach spaces. If every element in $X \widehat{\otimes}_\pi Y$ attains its projective norm, then the set of all bilinear forms on $X \times Y$ which attain their norms is dense in $\mathcal{B}(X \times Y)$. In other words, if $\text{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$, then

$$\overline{\text{NA}(X \times Y)}^{\|\cdot\|} = \mathcal{B}(X \times Y).$$

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Corollary

Let X, Y be Banach spaces. If $\text{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$, then

$$\overline{\text{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*).$$

Examples

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(a) If X is $L_1[0,1]$ and Y^* is a strictly convex Banach space without the Radon-Nikodým property, then the set $\text{NA}(L_1[0,1], Y^*)$ is not dense in $\mathcal{L}(L_1[0,1], Y^*)$.
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(b) There is a Banach space G such that $\mathrm{NA}(G \times \ell_p)$ is not dense in $\mathcal{B}(G \times \ell_p)$.
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(c) If X and Y are both $L_1[0,1]$, then the set $\mathrm{NA}(L_1[0,1] \times L_1[0,1])$ is not dense in $\mathcal{B}(L_1[0,1] \times L_1[0,1])$.
(Y.S. Choi, 1997)

Denseness of Nuclear Operators and Tensors Which attain their norms

The $\mathbf{L}_{o,o}$ (D., S.K. Kim, H.J. Lee, M. Mazzitelli)

Let X, Y and Z be Banach spaces. We say that $(X \times Y, Z)$ satisfies the $\mathbf{L}_{o,o}$ for bilinear mappings if given $\varepsilon > 0$ and $B \in \mathcal{B}(X \times Y, Z)$ with $\|B\| = 1$,

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there is $(x_0, y_0) \in S_X \times S_Y$ such that

$$\|B(x_0, y_0)\| = 1, \quad \|x - x_0\| < \varepsilon, \quad \text{and} \quad \|y - y_0\| < \varepsilon.$$

Examples of pairs which satisfy the $\mathbf{L}_{o,o}$
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(a) If $\dim(X), \dim(Y) < \infty$, then $(X \times Y, Z)$ has the $\mathbf{L}_{o,o}$ for every Banach space Z .

(b) $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear mappings if and only if (X, Y^*) has the $\mathbf{L}_{o,o}$ for operators, whenever Y is uniformly convex.

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(c) If $1 < p, q < \infty$, then $(\ell_p \times \ell_q, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ if and only if $p > q'$, where q' is the conjugate of q.

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(c) If $1 < p, q < \infty$, then $(\ell_p \times \ell_q, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ if and only if $p > q'$, where q' is the conjugate of q.

(d) There are reflexive X, Y such that $(X \times Y, \mathbb{K})$ fails the $\mathbf{L}_{o,o}$.

Theorem

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Corollary

Let X be finite dimensional Banach space.

(a) If Y is finite dimensional, then $\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_N} = \mathcal{N}(X,Y)$.

(b) If Y is uniformly convex, then $\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_N} = \mathcal{N}(X,Y)$.

Theorem

Let X, Y be Banach spaces. Suppose that $(X \times Y, \mathbb{K})$ has $\mathbf{L}_{o,o}$ for bilinear forms. Then,

$$\overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y.$$

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Let X be a finite dimensional Banach space.

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(b) If Y is uniformly convex, then $\overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y$.

Definition (Property (P))

Let X be a Banach space. We will say that X has the **property (P)** if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subseteq S_X$ a finite collection in the sphere, then we can find a finite dimensional subspace $M \subseteq X$ such that M is 1-complemented and there exists $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$ for every $i \in \{1, \ldots, n\}$.

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Observation

Property (P) is equivalent to the so-called metric π-property from P.G. Casazza's book on approximation properties.

The following Banach spaces satisfy property (P)

(a) Banach spaces with monotone Schauder basis.

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(e) $X = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$ or $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_p}$, $\forall\, 1 \leq p < \infty$, X_n satisfying property (P), $\forall n$.

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Theorem

Let X be a Banach space satisfying property (P) (or, equivalently, metric π-property).

(a) If Y satisfies property (P), then $\overline{\text{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$.

(b) If Y is uniformly convex, then $\overline{\text{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$.

Observation

If a Banach space Z has property (P), then it has the metric approximation property.

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Corollary

Let X be Banach space such that X^* satisfies property (P) (or, equivalently, metric π-property).

(a) If Y satisfies property (P), then $\overline{\text{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_N} = \mathcal{N}(X, Y)$.

(b) If Y is uniformly convex, then $\overline{\text{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_N} = \mathcal{N}(X, Y)$.

There are tensors which Cannot be Approximated By Norm-attaining tensors

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$\exists$ X and Y such that $\mathrm{NA}_\pi(X, Y^*)$ *is not* dense in $X \widehat{\otimes}_\pi Y^*$.

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(3) Try to guarantee that the set of operators which attain their norms is not bigger than the set of finite-rank operators.

Theorem

Let $\mathcal{R}$ be Read's space. There exists a subspace X of c_0 and Y of $\mathcal{R}$ such that the set of tensors in $X \widehat{\otimes}_\pi Y^*$ which attain their projective norms is not dense in $X \widehat{\otimes}_\pi Y^*$.

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(2) Let X, Y be reflexive Banach spaces. Is it true that the set of all norm-attaining tensors is dense in $X \widehat{\otimes}_\pi Y$?

(3) Are there Banach spaces X and Y so that $\text{NA}_{\mathcal{N}}(X, Y)$ is not dense in $\mathcal{N}(X, Y)$?

Further research on the topic
(Joint work with García-Lirola, M. Jung, A. Rueda Zoca)

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- Analogously...
  (1) $z \in \text{NA}_{\pi,s,N}(\widehat{\otimes}_{\pi,s,N}X)$.
  (2) $\exists P \in \mathcal{P}(^N X)$ with $\|P\| = 1$ such that $P(x_n) = 1, \forall n$.
  (3) $\forall P \in S_{\mathcal{P}(^N X)}$ such that $P(z) = \|z\|$ satisfies $P(x_n) = 1, \forall n$.

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- ...but that is another story...

Thank You
For Your Attention