Tensores (simétricos) e operadores nucleares que atingem a norma

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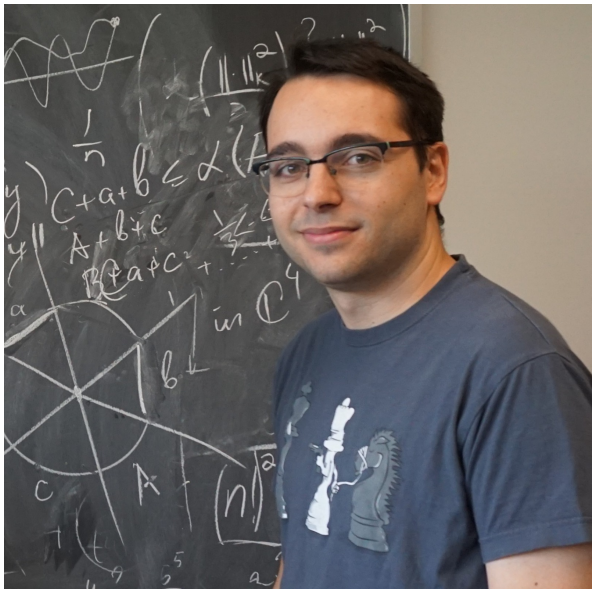

H.J. Lee, M. Mazzitelli, S.K. Kim

Sheldon Dantas **NA and BPBps**

Motivation

Definition

A functional $x^* \in X^*$ **attains the norm** if there is $x_0 \in S_X$ such that

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James Theorem

A Banach space X is reflexive if and only if every functional in X^* attains the norm.

Question

When is the set $\mathrm{NA}(X)$ dense in X^*?

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Is it true for bounded linear operators?

Definition

$T \in \mathcal{L}(X, Y)$ **attains the norm** if there is $x_0 \in S_X$ such that

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Bishop-Phelps' question

$\overline{\mathrm{NA}(X, Y)} = \mathcal{L}(X, Y)$ for every X, Y?

Lindenstrauss counterexample (1963)

There is a Banach space X such that

$$\overline{\mathrm{NA}(X,X)} \neq \mathcal{L}(X,X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators in general.

After this...

- Norm-attaining operators
  - J. Bourgain
  - R.E. Huff
  - W.T. Gowers
  - J. Johnson
  - W. Schachermayer
  - J.J. Uhl
  - J. Wolfe
  - V. Zizler
- Norm-attaining bilinear mappings
  - M. Acosta
  - R. Aron
  - F.J. Aguirre
  - Y.S. Choi
  - V. Lomonosov
  - R. Payá

After this...

- Norm-attaining homogeneous polynomials
  - D. Carando
  - D. García
  - S. Lassalle
  - M. Maestre
  - M. Mazzitelli
  - J.T. Rodríguez

More recently...

- B. Cascales
- R. Chiclana
- L.C. García-Lirola
- A. Guirao
- V. Kadets
- S.K. Kim
- M. Martín
- J. Merí
- V. Montesinos
- H.J. Lee
- G. López-Pérez
- D. Werner

Question (J. Diestel, J. Uhl, J. Johnson, J. Wolfe, $\approx$ 1970)

Can compact operators be approximated by norm-attaining ones?

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There exist compact operators between Banach spaces which **cannot** be approximated by norm-attaining operators.

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Main problem

Can finite-rank operators be approximated by norm-attaining ones?

Nuclear operators and Tensor Products

Projective tensor products

Given two Banach spaces X and Y, we denote by $X \widehat{\otimes}_\pi Y$ the projective tensor product of X and Y, which is defined as the completion of the normed space $X \otimes Y$ endowed with the norm

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$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

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- $\|x \otimes y\|_\pi = \|x\|\|y\|$,
- $B_{X\widehat{\otimes}_\pi Y}$ = closed convex hull of $B_X \otimes B_Y$, where $\{x \otimes y : x \in B_X, y \in B_Y\}$

Projective tensor products

- $(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$

under the action

$$G\left(\sum_{n=1}^{\infty} x_n \otimes y_n\right) = \sum_{n=1}^{\infty} G(x_n)(y_n)$$

for $G : X \longrightarrow Y^*$ as a linear functional on $X\widehat{\otimes}_\pi Y$.

Projective tensor products x Nuclear operators

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- There is a canonical operator $J : X^*\widehat{\otimes}_\pi Y \to \mathcal{L}(X, Y)$ with $\|J\| = 1$ such that

$$u = \sum_{n=1}^{\infty} x_n^* \otimes y_n \mapsto L_u,$$

where

$$L_u(x) := \sum_{n=1}^{\infty} x_n^*(x) y_n \quad (x \in X).$$

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The operators that arise in this way are called **nuclear operators**.

Nuclear operators

We denote by $\mathcal{N}(X, Y)$ the set of all nuclear operators with:

$$\|T\|_N := \inf\left\{\sum_{n=1}^{\infty}\|x_n^*\|\|y_n\| : T(x) = \sum_{n=1}^{\infty} x_n^*(x)y_n\right\},$$

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$\star$ If X^* or Y has the AP, then $X^*\widehat{\otimes}_\pi Y = \mathcal{N}(X, Y)$.

Norm-attainment Concepts

Norm-attaining definitions

(a) $z \in X \widehat{\otimes}_\pi Y$ **attains its projective norm** if there is a bounded sequence $(x_n, y_n) \subseteq X \times Y$ with $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$ such that $z = \sum_{n=1}^{\infty} x_n \otimes y_n$ and that $\|z\|_\pi = \sum_{n=1}^{\infty} \|x_n\| \|y_n\|$.

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(b) $T \in \mathcal{N}(X, Y)$ **attains its nuclear norm** if there is a bounded sequence $(x_n^*, y_n) \subseteq X^* \times Y$ with $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$ such that $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ and that $\|T\|_N = \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\|$.

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(d) $\mathrm{NA}_{\mathcal{N}}(X, Y) = \{T \in \mathcal{N}(X, Y) : T \text{ attains its nuclear norm}\}$.

Nuclear operators and Tensors Which attain their norms

Theorem

Let X, Y be Banach spaces. Let $z \in X \widehat{\otimes}_\pi Y$ with

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$.

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(1) $z \in \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$.

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Example (OneNote)

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Example (OneNote)

Proposition (J. Tomás Rodríguez)

Let X, Y be finite dimensional Banach spaces. Then,

$$\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y.$$

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Remark: Compare this with the classical theory.

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Remark: Compare this with the classical theory.

Proposition

Let H be a complex Hilbert space. Then,

(a) every nuclear operator $T \in \mathcal{N}(H, H)$ attains its nuclear norm.

(b) every tensor in $H \widehat{\otimes}_\pi H$ attains its projective norm.

It is natural to ask whether or not the equalities

$$\mathrm{NA}_{\mathcal{N}}(X,Y) = \mathcal{N}(X,Y) \quad \text{or} \quad \mathrm{NA}_{\pi}(X \widehat{\otimes}_{\pi} Y) = X \widehat{\otimes}_{\pi} Y$$

hold for every Banach spaces X and Y.

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Proposition

Let X, Y be Banach spaces. If every element in $X \widehat{\otimes}_\pi Y$ attains its projective norm, then the set of all bilinear forms on $X \times Y$ which attain their norms is dense in $\mathcal{B}(X \times Y)$. In other words, if $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$, then

$$\overline{\mathrm{NA}(X \times Y)}^{\|\cdot\|} = \mathcal{B}(X \times Y).$$

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Corollary

Let X, Y be Banach spaces. If $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$, then

$$\overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*).$$

Examples

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(a) If X is $L_1[0,1]$ and Y^* is a strictly convex Banach space without the Radon-Nikodým property, then the set $\mathrm{NA}(L_1[0,1], Y^*)$ is not dense in $\mathcal{L}(L_1[0,1], Y^*)$. (J.J. Uhl, 1976)

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(b) There is a Banach space G such that $\mathrm{NA}(G \times \ell_p)$ is not dense in $\mathcal{B}(G \times \ell_p)$. (W.T. Gowers, 1990)

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(b) There is a Banach space G such that $\mathrm{NA}(G \times \ell_p)$ is not dense in $\mathcal{B}(G \times \ell_p)$. (W.T. Gowers, 1990)

(c) If X and Y are both $L_1[0,1]$, then the set $\mathrm{NA}(L_1[0,1] \times L_1[0,1])$ is not dense in $\mathcal{B}(L_1[0,1] \times L_1[0,1])$. (Y.S. Choi, 1997)

Denseness of Nuclear Operators and Tensors Which attain their norms

The $\mathbf{L}_{o,o}$ (D., S.K. Kim, H.J. Lee, M. Mazzitelli)

Let X, Y and Z be Banach spaces. We say that $(X \times Y, Z)$ satisfies the $\mathbf{L}_{o,o}$ for bilinear mappings if given $\varepsilon > 0$ and $B \in \mathcal{B}(X \times Y, Z)$ with $\|B\| = 1$,

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there is $(x_0, y_0) \in S_X \times S_Y$ such that

$$\|B(x_0, y_0)\| = 1, \quad \|x - x_0\| < \varepsilon, \quad \text{and} \quad \|y - y_0\| < \varepsilon.$$

Examples of pairs which satisfy the $\mathbf{L}_{o,o}$
(D., S.K. Kim, H.J. Lee, M. Mazzitelli, 2020)

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Examples of pairs which satisfy the $\mathbf{L}_{o,o}$
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(a) If $\dim(X), \dim(Y) < \infty$, then $(X \times Y, Z)$ has the $\mathbf{L}_{o,o}$ for every Banach space Z .

(b) $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear mappings if and only if (X, Y^*) has the $\mathbf{L}_{o,o}$ for operators, whenever Y is uniformly convex.

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(b) There are reflexive X, Y such that $(X \times Y, \mathbb{K})$ fails the $\mathbf{L}_{o,o}$.

Theorem

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Let X be finite dimensional Banach space. If Y is uniformly convex, then

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Theorem

Let X, Y be Banach spaces. Suppose that $(X \times Y, \mathbb{K})$ has $\mathbf{L}_{o,o}$ for bilinear forms. Then,

$$\overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y.$$

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Observation

The metric π-property is defined in P.G. Casazza's book on approximation properties.

The following Banach spaces satisfy the metric π-property

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(e) $X = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$ or $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_p}$, $\forall\ 1 \leq p < \infty$, X_n satisfying the metric π-property, $\forall n$.

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Theorem

Let X be a Banach space satisfying the metric π-property. If Y satisfies the metric π-property or it is uniformly convex, then

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Observation

If a Banach space Z has metric π-property, then it has the metric approximation property.

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Corollary

Let X be Banach space such that X^* satisfies the metric π-property. If Y satisfies the metric π-property or it is uniformly convex, then

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Theorem

Let Y be a dual space. Then

$$\overline{\mathrm{NA}_\pi(c_0 \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = c_0 \widehat{\otimes}_\pi Y.$$

Theorem

If X^*, Y^* have the RNP, then

$$\overline{\mathrm{NA}_{\mathcal{N}}(X, Y^*)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X, Y^*).$$

If X^*, Y^* have the RNP and at least one of them has the AP, then

$$\overline{\mathrm{NA}_\pi(X^* \widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} = X^* \widehat{\otimes}_\pi Y^*.$$

There are tensors which Cannot be Approximated By Norm-attaining tensors

Idea

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(3) Try to guarantee that the set of operators which attain their norms is not bigger than the set of finite-rank operators.

Theorem

Let $\mathcal{R}$ be Read's space. There exists a subspace X of c_0 and Y of $\mathcal{R}$ such that the set of tensors in $X\widehat{\otimes}_\pi Y^*$ which attain their projective norms is not dense in $X\widehat{\otimes}_\pi Y^*$.

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Further research on the topic
(Joint work with García-Lirola, M. Jung, A. Rueda Zoca)

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- As a counterpart from the first part...
  (1) $z \in \mathrm{NA}_{\pi,s,N}(\widehat{\otimes}_{\pi,s,N}X)$
  (2) $\exists P \in \mathcal{P}(^N X)$ with $\|P\| = 1$ such that $P(x_n) = 1, \forall n$
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(4) If X^* has RNP, then is it true that $\mathrm{NA}_{\mathcal{N}}(X, Y^*)$ is dense in $\mathcal{N}(X, Y^*)$ for **every** Banach space Y?

Thank You
For Your Attention