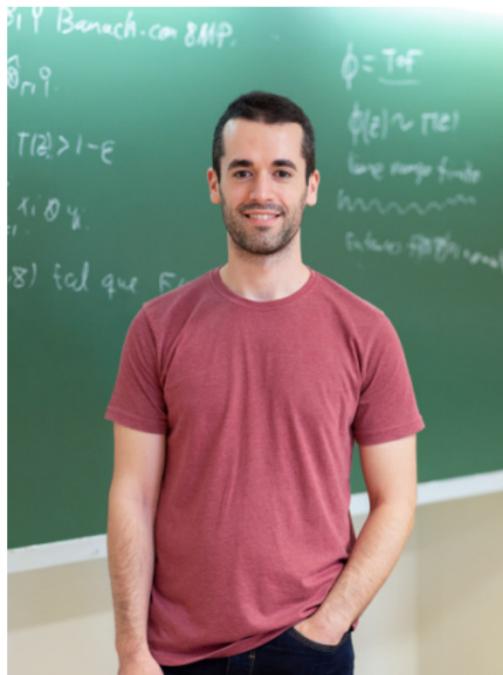# A characterization of a local vector valued Bollobás theorem

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X, Y and Z are real or complex Banach spaces.

- $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$,
- B_X is the closed unit ball of X,
- S_X is the unit sphere of X,
- $\mathcal{L}(X, Y)$ continuous linear operators from X into Y,
- $\mathcal{K}(X, Y)$ compact linear operators from X into Y,
- $X^* = \mathcal{L}(X; \mathbb{K})$ topological dual of X.

Definition

$x^* \in X^*$ **attains the norm** if there is $x_0 \in S_X$ such that

$$|x^*(x_0)| = \|x^*\| = \sup_{x \in S_X} |x^*(x)|.$$

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James Theorem

X is reflexive if and only if every functional in X^* attains the norm.

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Question

Is it true for bounded linear operators?

Definition

$T \in \mathcal{L}(X, Y)$ **attains the norm** if there is $x_0 \in S_X$ such that

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Bishop-Phelps' question

$\overline{\mathsf{NA}(X, Y)} = \mathcal{L}(X, Y)$ for every X, Y?

There is a Banach space X such that

$$\overline{\mathsf{NA}(X, X)} \neq \mathcal{L}(X, X),$$

showing that the Bishop-Phelps result **does not** hold for bounded linear operators in general.

After this...

- Norm-attaining operators

  - J. Bourgain
  - R.E. Huff
  - W.T. Gowers
  - J. Johnson

  - W. Schachermayer
  - J.J. Uhl
  - J. Wolfe
  - V. Zizler

- Norm-attaining bilinear mappings

  - M. Acosta
  - R. Aron
  - F.J. Aguirre

  - Y.S. Choi
  - V. Lomonosov
  - R. Payá

More recently...

- B. Cascales
- D. Carando
- D. García
- H.J. Lee

- S. Lassalle
- M. Maestre
- M. Martín
- V. Montesinos

And even more...

- G. Martínez-Cervantes
- R. Chiclana
- J. Falcó
- L.C. García-Lirola
- A. Guirao
- V. Kadets
- S.K. Kim

- M. Jung
- M. Mazzitelli
- J. Merí
- G. López-Pérez
- Ó. Roldán
- A. Rueda Zoca
- D. Werner

Question (J. Diestel, J. Uhl, J. Johnson, J. Wolfe, $\approx$ 1970)

Can compact operators be approximated by norm-attaining ones?

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There exist compact operators between Banach spaces which **cannot** be approximated by norm-attaining operators.

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Answer (M. Martín, JFA, 2014)

There exist compact operators between Banach spaces which **cannot** be approximated by norm-attaining operators.

Main problem

Can finite-rank operators be approximated by norm-attaining ones?

Bollobás Theorem

In 1970, Bollobás improved the Bishop-Phelps theorem.

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Let $\varepsilon \in (0, 2)$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2},$$

there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|y^*\| = |y^*(y)| = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

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(2014, M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido)

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- Bollobás theorem does make sense in the reflexive setting.

- It seems to be natural to wonder whether there is a version of Bollobás theorem *without* changing the initial functional x^*.

Stronger version of the Bollobás theorem?

Given $x \in S_X$ and $x^* \in S_{X^*}$ with

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

is it possible to guarantee the existence of $y \in S_X$ such that

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Yes. When X is uniformly convex. **PROVE IT!**
In fact, this is a characterization for uniformly convex spaces
(Kim, Lee, 2014)

In a more general scenario, we wonder:

Given $\varepsilon > 0$, is it possible to find $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

one can find a new element $x_0 \in S_X$ such that

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No!

All operators $T : X \to Y$ fail such a property with X, Y of dim ≥ 2.

(D., Kadets, Kim, Lee, Martín, 2018)

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★ Chakraborty, 2021
★ D., 2017
★ D., Kim, Kee, Mazzitelli, 2017
★ D., Kim, Kee, Mazzitelli, 2020
★ Sain, 2019
★ Talponen, 2017

Property ⋆

(X, Y) has property ⋆ for operators if given $\varepsilon > 0$ and $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, T),$$

there is $x_0 \in S_X$ such that

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- If a reflexive X satisfies Kadec-Klee, then (X, Y) has property $\star$ for compact operators for every Y.
  [Sain, 2019]

Theorem A

Let X be a reflexive Banach space. TFAE:

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- X^* is Fréchet iff X is stric. conv. and $(X, \mathbb{K})$ has property $\star$. [D., Kim, Lee, Mazzitelli, 2018]

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$\Rightarrow$ *Theorem A generalizes Talponen and Sain's results*

- Whenever X is strictly convex and Y is arbitrary, (X, Y) has property $\star$ for compact operators iff X^* is Fréchet differentiable. [Talponen, 2017]

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REFLEXIVITY OF THE PROJECTIVE TENSOR PRODUCT $X \widehat{\otimes}_\pi Y$

Projective tensor products

Given two Banach spaces X and Y, we denote by $X\widehat{\otimes}_\pi Y$ the projective tensor product of X and Y, which is defined as the completion of the normed space $X \otimes Y$ endowed with the norm

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where the infimum is taken over all representation of z of the form $z = \sum_{i=1}^n x_i \otimes y_i$.

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- $\|x \otimes y\|_\pi = \|x\| \|y\|$,
- $B_{X \widehat{\otimes}_\pi Y}$ = closed convex hull of $B_X \otimes B_Y$, where $\{x \otimes y : x \in B_X, y \in B_Y\}$

Dual of $X \widehat{\otimes}_\pi Y$

- $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$

under the action

$$G\left(\sum_{n=1}^{\infty} x_n \otimes y_n\right) = \sum_{n=1}^{\infty} G(x_n)(y_n)$$

for an operator $G : X \to Y^*$ as a linear functional on $X \widehat{\otimes}_\pi Y$.

Dual of $X\widehat{\otimes}_\pi Y$

- $(X\widehat{\otimes}_\pi Y)^* = \mathcal{B}(X \times Y; \mathbb{K})$

under the action

$$B\left(\sum_{n=1}^\infty x_n \otimes y_n\right) = \sum_{n=1}^\infty B(x_n, y_n)$$

for a bilinear form $B : X \times Y \to \mathbb{K}$ as a linear functional on $X\widehat{\otimes}_\pi Y$.

Property ⋆ for bilinear forms

$(X \times Y; \mathbb{K})$ has property ⋆ for bilinear forms if given $\varepsilon > 0$ and $B \in \mathcal{B}(X \times Y; \mathbb{K})$ with $\|B\| = 1$, there is $\eta(\varepsilon, B) > 0$ such that whenever $(x, y) \in S_X \times S_Y$ satisfies

$$|B(x, y)| > 1 - \eta(\varepsilon, B),$$

there is $(x_0, y_0) \in S_X \times S_Y$ such that

$$|B(x_0, y_0)| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|y_0 - y\| < \varepsilon.$$

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- $X = \ell_p$ and $Y = \ell_q$ iff $p > q'$, $q' =$ conjugate index of q

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- One might think that the reflexivity of $X \widehat{\otimes}_\pi Y$ plays an important role here:

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Theorem (consequence of Pitt's theorem)

$\ell_p \widehat{\otimes}_\pi \ell_q$ is reflexive iff $p > q'$

Theorem B

Let X be a strictly convex Banach space or a Banach space satisfying the Kadec-Klee property.

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(c) $(X \times Y, \mathbb{K})$ has property $\star$ for bilinear forms.

(d) $(X \widehat{\otimes}_\pi Y, \mathbb{K})$ has property $\star$ for linear functionals.

Corollary of Theorem B

Let $1 < p, q < \infty$ and let q' be the conjugate of q.

(a) $(\ell_p \times \ell_q; \mathbb{K})$ **has** property $\star$ for bilinear forms iff $p > q'$.

(b) $(L_p(\mu), L_q(\mu); \mathbb{K})$ **fails** property $\star$ for bilinear forms for not purely atomic measures μ, ν.

WHAT IS IT FOR AND WHAT NEXT?

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  [D., Jung, Roldán, Rueda Zoca], 2020]
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- $(X, \mathbb{K})$ has property $\star$ for linear functionals iff X^* is SSD (strongly subdifferentiable). Study a dual version of property $\star$ in order to study the strong subdifferentiability of $X \widehat{\otimes}_\pi Y$.

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$$\textit{When } X \widehat{\otimes}_\pi Y \textit{ is SSD?}$$

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  When $X \widehat{\otimes}_\pi Y$ is SSD?

  When $\widehat{\otimes}_{\pi,s,N} X$ is SSD?

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  [D., Kim, Lee, Mazzitelli, 2020]
  [D., Jung, Mazzitelli, Tomás-Rodríguez, -]

Thank You
For Your Attention