

On the strong subdifferentiability of the homogeneous polynomials and (symmetric) tensor products

SHELDON DANTAS

UNIVERSITAT JAUME I
DEPARTAMENT DE MATEMÀTIQUES I
INSTITUT DE MATEMÀTIQUES I APLICACIONS DE CASTELLÓN (IMAC)

Congreso Bial de la Real Sociedad Matemática Española
RSME2022 - Ciudad Real

Based on a few joint works with

- ★ Sun Kwang Kim (Chungbuk National University)
- ★ Han Ju Lee (Dongguk University)

- ★ Abraham Rueda Zoca (Universidad de Murcia)

- ★ Mingu Jung (Pohang University of Sciences and Technology)
- ★ Martin Mazzitelli (Universidad Nacional de Cuyo)
- ★ Jorge Tomás Rodríguez (Universidad N.C.P. Buenos Aires)

Strong subdifferentiability of the norm

Strong subdifferentiability of the norm

We say that the norm of a Banach space X is **strongly subdifferentiable** (**SSD**, for short) at a point $u \in S_X$ if the one-sided limit

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (\|u + tx\| - 1)$$

exists uniformly for $x \in B_X$.

Strong subdifferentiability of the norm

We say that the norm of a Banach space X is **strongly subdifferentiable** (**SSD**, for short) at a point $u \in S_X$ if the one-sided limit

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (\|u + tx\| - 1)$$

exists uniformly for $x \in B_X$.

- ★ The norm is Fréchet differentiable iff it is Gâteaux and SSD.

Strong subdifferentiability of the norm

We say that the norm of a Banach space X is **strongly subdifferentiable** (**SSD**, for short) at a point $u \in S_X$ if the one-sided limit

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (\|u + tx\| - 1)$$

exists uniformly for $x \in B_X$.

- ★ The norm is Fréchet differentiable iff it is Gâteaux and SSD.
- ★ Consider ϕ_n on B_X defined by

$$\phi_n(x) = \frac{1}{n} \left(\left\| u + \frac{x}{n} \right\| - 1 \right) = \|nu + x\| - n.$$

Strong subdifferentiability of the norm

We say that the norm of a Banach space X is **strongly subdifferentiable** (**SSD**, for short) at a point $u \in S_X$ if the one-sided limit

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (\|u + tx\| - 1)$$

exists uniformly for $x \in B_X$.

- ★ The norm is Fréchet differentiable iff it is Gâteaux and SSD.
- ★ Consider ϕ_n on B_X defined by

$$\phi_n(x) = \frac{1}{n} \left(\left\| u + \frac{x}{n} \right\| - 1 \right) = \|nu + x\| - n.$$

- ★ Then, the norm of X is SSD iff $\{\phi_n\}$ converges uniformly on B_X .

Strong subdifferentiability of the norm

Theorem (C. Franchetti and R. Payá, 1993)

The following are equivalent.

- ★ u strongly exposes the set $D(u)$.
- ★ D is $(n - n)$ upper semicontinuous at u .
- ★ For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that

$$\text{dist}(D(x), D(y)) = \inf \left\{ \|g - f\| : g \in D(x), f \in D(u) \right\} < \varepsilon$$

whenever $x \in S_X$ satisfies $\|x - u\| < \delta$.

- ★ The norm of X is SSD at u .
- ★ u is a τ -point of X .

Strong subdifferentiability of the norm

- ★ The norm of any finite-dimensional space is SSD.
(Dini's theorem)

Strong subdifferentiability of the norm

- ★ The norm of any finite-dimensional space is SSD.
(Dini's theorem)
- ★ The sup-norm on c_0 is SSD at every point of c_0 .
(C. Franchetti, 1986)

Strong subdifferentiability of the norm

- ★ The norm of any finite-dimensional space is SSD.
(Dini's theorem)
- ★ The sup-norm on c_0 is SSD at every point of c_0 .
(C. Franchetti, 1986)
- ★ The set of all SSD points of the sup-norm of ℓ_∞ is not a G_δ in ℓ_∞ .
(G. Godefroy, V. Montesinos, V. Zizler, 1995)

Strong subdifferentiability of the norm

- ★ The norm of any finite-dimensional space is SSD.
(Dini's theorem)
- ★ The sup-norm on c_0 is SSD at every point of c_0 .
(C. Franchetti, 1986)
- ★ The set of all SSD points of the sup-norm of ℓ_∞ is not a G_δ in ℓ_∞ .
(G. Godefroy, V. Montesinos, V. Zizler, 1995)
- ★ The norm of ℓ_1 is only SSD at points in S_{ℓ_1} which are sequences with finitely many nonzero terms.
(J.R. Giles, D A. Gregory, B. Sims, 1978)

Strong subdifferentiability of the norm

- ★ The norm of any finite-dimensional space is SSD.
(Dini's theorem)
- ★ The sup-norm on c_0 is SSD at every point of c_0 .
(C. Franchetti, 1986)
- ★ The set of all SSD points of the sup-norm of ℓ_∞ is not a G_δ in ℓ_∞ .
(G. Godefroy, V. Montesinos, V. Zizler, 1995)
- ★ The norm of ℓ_1 is only SSD at points in S_{ℓ_1} which are sequences with finitely many nonzero terms.
(J.R. Giles, D A. Gregory, B. Sims, 1978)
- ★ The norm of X is SSD when X is a predual of the
 - ★ Hardy space H^1 of analytic functions on the ball,
 - ★ Lorentz spaces $L_{p,1}(\mu)$,
 - ★ Trace Class \mathcal{C}_1 .(S.J. Dilworth, D. Kutzarova, 1995)

Strong subdifferentiability of the norm

- ★ A Banach space with an SSD norm is Asplund.
(C. Franchetti and R. Payá, 1993)
(G. Godefroy, V. Montesinos, V. Zizler, 1995)

Strong subdifferentiability of the norm

PROBLEM

When are the norms of

$$X \widehat{\otimes}_{\pi} Y \quad X \widehat{\otimes}_{\varepsilon} Y \quad \widehat{\otimes}_{\pi, s, N} X \quad \mathcal{L}(X, Y) \quad \mathcal{P}(^N X, Y)$$

strongly subdifferentiable?

Strong subdifferentiability of the norm

PROBLEM

When are the norms of

$$X \widehat{\otimes}_{\pi} Y$$

$$X \widehat{\otimes}_{\varepsilon} Y$$

$$\widehat{\otimes}_{\pi, s, N} X$$

$$\mathcal{L}(X, Y)$$

$$\mathcal{P}(^N X, Y)$$

strongly subdifferentiable?

Strong subdifferentiability of the norm

The **projective tensor product** $X \widehat{\otimes}_{\pi} Y$ is defined as the completion of the algebraic tensor product $X \otimes Y$ endowed with the norm

$$\|z\|_{\pi} := \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i \right\}$$

where the infimum is taken over all representations of z of the form $z = \sum_{i=1}^n x_i \otimes y_i$.

Strong subdifferentiability of the norm

The **projective tensor product** $X \widehat{\otimes}_{\pi} Y$ is defined as the completion of the algebraic tensor product $X \otimes Y$ endowed with the norm

$$\|z\|_{\pi} := \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i \right\}$$

where the infimum is taken over all representations of z of the form $z = \sum_{i=1}^n x_i \otimes y_i$.

- ★ $B_{X \widehat{\otimes}_{\pi} Y} = \overline{\text{co}}(B_X \otimes B_Y)$
- ★ $(X \widehat{\otimes}_{\pi} Y)^* = \mathcal{B}(X \times Y, \mathbb{K})$

Strong subdifferentiability of the norm

The **projective symmetric tensor product** $\widehat{\otimes}_{\pi,s,N} X$ is the completion of the linear space $\otimes_{\pi,s,N} X$ generated by $\{\otimes^N z : z \in X\}$ endowed with the norm

$$\|z\|_{\pi,s,N} := \inf \left\{ \sum_{i=1}^n |\lambda_i| : z = \sum_{i=1}^n \lambda_i \otimes^N x_i \right\}$$

where the infimum is taken over all the possible representations of z of that form.

Strong subdifferentiability of the norm

The **projective symmetric tensor product** $\widehat{\otimes}_{\pi,s,N} X$ is the completion of the linear space $\otimes_{\pi,s,N} X$ generated by $\{\otimes^N z : z \in X\}$ endowed with the norm

$$\|z\|_{\pi,s,N} := \inf \left\{ \sum_{i=1}^n |\lambda_i| : z = \sum_{i=1}^n \lambda_i \otimes^N x_i \right\}$$

where the infimum is taken over all the possible representations of z of that form.

- ★ $B_{\widehat{\otimes}_{\pi,s,N} X} = \overline{\text{aco}}(\{\otimes^N x : x \in S_X\})$
- ★ $(\widehat{\otimes}_{\pi,s,N} X)^* = \mathcal{P}(^N X, \mathbb{K})$

Strong subdifferentiability of the norm

PROBLEM

When are the norms of

$$X \widehat{\otimes}_{\pi} Y$$

$$X \widehat{\otimes}_{\varepsilon} Y$$

$$\widehat{\otimes}_{\pi, s, N} X$$

$$\mathcal{L}(X, Y)$$

$$\mathcal{P}({}^N X, Y)$$

strongly subdifferentiable?

Strong subdifferentiability of the norm

Question

Which characterization of the SSD should we use to tackle this problem?

Strong subdifferentiability of the norm

Question

Which characterization of the SSD should we use to tackle this problem?

- ★ The norm $\|\cdot\|$ of a Banach space X is SSD at $x \in S_X$ if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon, x) > 0$ such that whenever $x^* \in S_{X^*}$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x)$, there exists $y^* \in S_{X^*}$ such that $|y^*(x)| = 1$ and $\|y^* - x^*\| < \varepsilon$.

(C. Franchetti and R. Payá, 1993)

(G. Godefroy, V. Montesinos, V. Zizler, 1995)

Strong subdifferentiability of the norm

Question

Which characterization of the SSD should we use to tackle this problem?

- ★ The norm $\|\cdot\|$ of a Banach space X is SSD at $x \in S_X$ if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon, x) > 0$ such that whenever $x^* \in S_{X^*}$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x)$, there exists $y^* \in S_{X^*}$ such that $|y^*(x)| = 1$ and $\|y^* - x^*\| < \varepsilon$.
(C. Franchetti and R. Payá, 1993)
(G. Godefroy, V. Montesinos, V. Zizler, 1995)
- ★ **Dual version:** Given $\varepsilon > 0$ and $x^* \in S_{X^*}$, there exists $\eta(\varepsilon, x^*) > 0$ such that whenever $x \in S_X$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x^*)$, there exists $y \in S_X$ such that $|x^*(y)| = 1$ and $\|y - x\| < \varepsilon$.

Strong subdifferentiability of the norm

Question

Which characterization of the SSD should we use to tackle this problem?

- ★ The norm $\|\cdot\|$ of a Banach space X is SSD at $x \in S_X$ if and only if **(C1)** given $\varepsilon > 0$, there exists $\eta(\varepsilon, x) > 0$ such that whenever $x^* \in S_{X^*}$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x)$, there exists $y^* \in S_{X^*}$ such that $|y^*(x)| = 1$ and $\|y^* - x^*\| < \varepsilon$.
- ★ **Dual version: (C2)** Given $\varepsilon > 0$ and $x^* \in S_{X^*}$, there exists $\eta(\varepsilon, x^*) > 0$ such that whenever $x \in S_X$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x^*)$, there exists $y \in S_X$ such that $|x^*(y)| = 1$ and $\|y - x\| < \varepsilon$.

Strong subdifferentiability of $l_p \widehat{\otimes}_\pi l_q$

Property (C2) was used successively **as a tool** recently.

Theorem (S.D. and A. Rueda Zoca, 2021)

The norm of $l_p \widehat{\otimes}_\pi l_q$ is SSD if and only if $p^{-1} + q^{-1} < 1$.

Strong subdifferentiability on
 $\mathcal{P}({}^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

SSD on $\mathcal{P}(^N X, Y^*)$

(S.D., Jung, Mazzitelli, Rodríguez, 2022) Consider the following.

(A) The norm of $\mathcal{P}(^N X, Y^*)$ is SSD.

SSD on $\mathcal{P}(^N X, Y^*)$

(S.D., Jung, Mazzitelli, Rodríguez, 2022) Consider the following.

(A) The norm of $\mathcal{P}(^N X, Y^*)$ is SSD.

(B) $((\widehat{\otimes}_{\pi, s, N} X) \widehat{\otimes}_{\pi} Y, \mathbb{K})$ has (C2) for linear functionals.

SSD on $\mathcal{P}(^N X, Y^*)$

(S.D., Jung, Mazzitelli, Rodríguez, 2022) Consider the following.

(A) The norm of $\mathcal{P}(^N X, Y^*)$ is SSD.

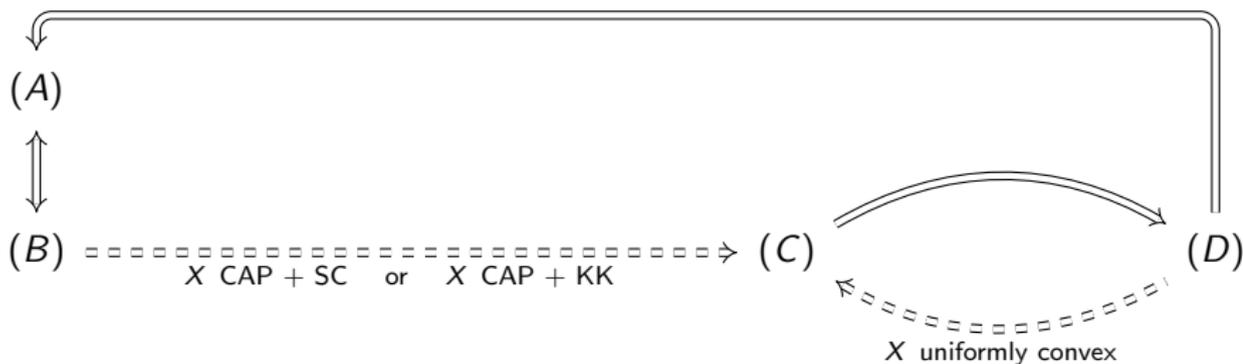
(B) $((\widehat{\otimes}_{\pi,s,N} X) \widehat{\otimes}_{\pi} Y, \mathbb{K})$ has (C2) for linear functionals.

(C) (X, Y^*) has N -(C2).

SSD on $\mathcal{P}(^N X, Y^*)$

(S.D., Jung, Mazzitelli, Rodríguez, 2022) Consider the following.

- (A) The norm of $\mathcal{P}(^N X, Y^*)$ is SSD.
- (B) $((\widehat{\otimes}_{\pi, s, N} X) \widehat{\otimes}_{\pi} Y, \mathbb{K})$ has (C2) for linear functionals.
- (C) (X, Y^*) has N -(C2).
- (D) (X, Y^*) has property \star .



SSD on $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

(S.D., Jung, Mazzitelli, Rodríguez, 2022) Consider the following.

(A2) The norm of $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$ is SSD.

(B2) The norm of $\mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*)$ is SSD.

(C2) $(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N, \mathbb{K})$ has (C2) for linear functionals.

(D2) $(X_1, \dots, X_N, \mathbb{K})$ has the M -(C2).

SSD on $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

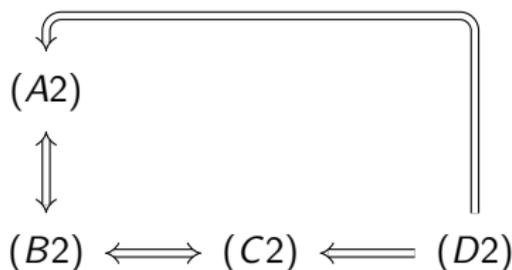
(S.D., Jung, Mazzitelli, Rodríguez, 2022) Consider the following.

(A2) The norm of $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$ is SSD.

(B2) The norm of $\mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*)$ is SSD.

(C2) $(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N, \mathbb{K})$ has (C2) for linear functionals.

(D2) $(X_1, \dots, X_N, \mathbb{K})$ has the M -(C2).



SSD on $\mathcal{P}(^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

Corollary (S.D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p, q < \infty$ and $N \in \mathbb{N}$. The norm of

(a) $\mathcal{P}(^N \ell_p, \mathbb{K})$ is SSD if and only if $N < p$.

SSD on $\mathcal{P}(^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

Corollary (S.D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p, q < \infty$ and $N \in \mathbb{N}$. The norm of

(a) $\mathcal{P}(^N \ell_p, \mathbb{K})$ is SSD if and only if $N < p$.

(b) $\mathcal{P}(^N \ell_p, \ell_q)$ is SSD if and only if $Nq < p$.

SSD on $\mathcal{P}(^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

Corollary (S.D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p, q < \infty$ and $N \in \mathbb{N}$. The norm of

(a) $\mathcal{P}(^N \ell_p, \mathbb{K})$ is SSD if and only if $N < p$.

(b) $\mathcal{P}(^N \ell_p, \ell_q)$ is SSD if and only if $Nq < p$.

(c) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \mathbb{K})$ is SSD if and only if $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.

SSD on $\mathcal{P}(^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

Corollary (S.D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p, q < \infty$ and $N \in \mathbb{N}$. The norm of

(a) $\mathcal{P}(^N \ell_p, \mathbb{K})$ is SSD if and only if $N < p$.

(b) $\mathcal{P}(^N \ell_p, \ell_q)$ is SSD if and only if $Nq < p$.

(c) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \mathbb{K})$ is SSD if and only if $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.

(d) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \ell_q)$ is SSD if and only if $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}$.

Exactly when these spaces are reflexive.

(Uniform) strong subdifferentiability on
 $\widehat{\otimes}_{\pi, s, N} X$ and $X \widehat{\otimes}_{\pi} Y$

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Definition

The norm of a Banach space X is **uniformly strongly subdifferentiable** on $U \subseteq S_X$ whenever the limit

$$\lim_{t \rightarrow 0^+} \frac{\|u + tz\| - 1}{t} = \max \left\{ \operatorname{Re} x^*(z) : x^* \in D(u) \right\} = \tau(u, z)$$

exists uniformly for $(u, z) \in U \times B_X$.

where

$$D(u) = \left\{ x^* \in X^* : \|x^*\| = x^*(u) = 1 \right\}.$$

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Equivalently,

(C3) Uniformly SSD on $U \subseteq S_X$

The norm of X is uniformly SSD on $U \subseteq S_X$ if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x_0 \in U$ satisfy

$$|x^*(x_0)| > 1 - \eta(\varepsilon),$$

there exists a new functional $y^* \in S_{X^*}$ such that

$$|y^*(x_0)| = 1 \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

(Prove it!)

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$.

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$.

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$.

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**,

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that $\operatorname{Re} \varphi(u) > 1 - \eta(\varepsilon)$

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that $\operatorname{Re} \varphi(u) > 1 - \eta(\varepsilon)$ and such that $\|\varphi - \tilde{\varphi}\| \geq \varepsilon$ for every $\tilde{\varphi} \in S_{X^*}$ with $\tilde{\varphi}(u) = 1$.

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that $\operatorname{Re} \varphi(u) > 1 - \eta(\varepsilon)$ and such that $\|\varphi - \tilde{\varphi}\| \geq \varepsilon$ for every $\tilde{\varphi} \in S_{X^*}$ with $\tilde{\varphi}(u) = 1$. Then, $D(u)$ and $\varphi + \varepsilon B_{X^*}$ are w^* -compact, convex, and disjoint sets.

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that $\operatorname{Re} \varphi(u) > 1 - \eta(\varepsilon)$ and such that $\|\varphi - \tilde{\varphi}\| \geq \varepsilon$ for every $\tilde{\varphi} \in S_{X^*}$ with $\tilde{\varphi}(u) = 1$. Then, $D(u)$ and $\varphi + \varepsilon B_{X^*}$ are w^* -compact, convex, and disjoint sets. By the HB separation theorem

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that $\operatorname{Re} \varphi(u) > 1 - \eta(\varepsilon)$ and such that $\|\varphi - \tilde{\varphi}\| \geq \varepsilon$ for every $\tilde{\varphi} \in S_{X^*}$ with $\tilde{\varphi}(u) = 1$. Then, $D(u)$ and $\varphi + \varepsilon B_{X^*}$ are w^* -compact, convex, and disjoint sets. By the HB separation theorem, there exists $z \in S_X$ such that

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that $\operatorname{Re} \varphi(u) > 1 - \eta(\varepsilon)$ and such that $\|\varphi - \tilde{\varphi}\| \geq \varepsilon$ for every $\tilde{\varphi} \in S_{X^*}$ with $\tilde{\varphi}(u) = 1$. Then, $D(u)$ and $\varphi + \varepsilon B_{X^*}$ are w^* -compact, convex, and disjoint sets. By the HB separation theorem, there exists $z \in S_X$ such that

$$\begin{aligned} \tau(u, z) = \max\{\operatorname{Re} \tilde{\varphi}(z) : \tilde{\varphi} \in D(u)\} &\leq \min\{\operatorname{Re}(\varphi + \varepsilon\psi)(z) : \psi \in B_{X^*}\} \\ &= \operatorname{Re} \varphi(z) - \varepsilon. \end{aligned}$$

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that $\operatorname{Re} \varphi(u) > 1 - \eta(\varepsilon)$ and such that $\|\varphi - \tilde{\varphi}\| \geq \varepsilon$ for every $\tilde{\varphi} \in S_{X^*}$ with $\tilde{\varphi}(u) = 1$. Then, $D(u)$ and $\varphi + \varepsilon B_{X^*}$ are w^* -compact, convex, and disjoint sets. By the HB separation theorem, there exists $z \in S_X$ such that

$$\begin{aligned} \tau(u, z) = \max\{\operatorname{Re} \tilde{\varphi}(z) : \tilde{\varphi} \in D(u)\} &\leq \min\{\operatorname{Re}(\varphi + \varepsilon\psi)(z) : \psi \in B_{X^*}\} \\ &= \operatorname{Re} \varphi(z) - \varepsilon. \end{aligned}$$

Then, for $t = \frac{\delta}{2}$, we have

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

Sketch of the proof: Suppose that the norm of X is uniformly SSD on $U \subseteq S_X$. Let $\delta > 0$. If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$. If (C3) is **false**, then there exist $u \in U$ and norm-one $\varphi \in S_{X^*}$ such that $\operatorname{Re} \varphi(u) > 1 - \eta(\varepsilon)$ and such that $\|\varphi - \tilde{\varphi}\| \geq \varepsilon$ for every $\tilde{\varphi} \in S_{X^*}$ with $\tilde{\varphi}(u) = 1$. Then, $D(u)$ and $\varphi + \varepsilon B_{X^*}$ are w^* -compact, convex, and disjoint sets. By the HB separation theorem, there exists $z \in S_X$ such that

$$\begin{aligned} \tau(u, z) = \max\{\operatorname{Re} \tilde{\varphi}(z) : \tilde{\varphi} \in D(u)\} &\leq \min\{\operatorname{Re}(\varphi + \varepsilon\psi)(z) : \psi \in B_{X^*}\} \\ &= \operatorname{Re} \varphi(z) - \varepsilon. \end{aligned}$$

Then, for $t = \frac{\delta}{2}$, we have

$$\frac{\varepsilon}{2} > \frac{\|u + tz\| - 1}{t} - \tau(u, z) \geq \frac{\operatorname{Re} \varphi(u + tz) - 1}{t} - \operatorname{Re} \varphi(z) + \varepsilon \geq \dots \geq \frac{\varepsilon}{2}.$$

(Uniformly) SSD on $\widehat{\otimes}_{\pi, S, N} X$ and $X \widehat{\otimes}_{\pi} Y$

(C3) Uniformly SSD on $U \subseteq S_X$

The norm of X is uniformly SSD on $U \subseteq S_X$ if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x_0 \in U$ satisfy

$$|x^*(x_0)| > 1 - \eta(\varepsilon),$$

there exists a new functional $y^* \in S_{X^*}$ such that

$$|y^*(x_0)| = 1 \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

Consider the following subsets:

Consider the following subsets:

$$U = \left\{ x_1 \otimes \cdots \otimes x_N : \|x_j\| = 1 \right\} \subseteq S_{X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N},$$

$$U_s := \left\{ \otimes^N x : \|x\| = 1 \right\} \subseteq S_{\hat{\otimes}_{\pi,s,N} X}.$$

Consider the following subsets:

$$U = \left\{ x_1 \otimes \cdots \otimes x_N : \|x_j\| = 1 \right\} \subseteq S_{X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N},$$

$$U_s := \left\{ \otimes^N x : \|x\| = 1 \right\} \subseteq S_{\hat{\otimes}_{\pi,s,N} X}.$$

Theorem (S.D., Jung, Mazzitelli, Rodríguez, 2022)

(a) The projective symmetric norm of

(1) $\hat{\otimes}_{\pi,s,N} \ell_2$ is uniformly SSD on U_s .

(2) $\hat{\otimes}_{\pi,s,N} c_0$ is SSD on U_s (in the complex case).

Consider the following subsets:

$$U = \left\{ x_1 \otimes \cdots \otimes x_N : \|x_j\| = 1 \right\} \subseteq S_{X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N},$$

$$U_s := \left\{ \otimes^N x : \|x\| = 1 \right\} \subseteq S_{\hat{\otimes}_{\pi,s,N} X}.$$

Theorem (S.D., Jung, Mazzitelli, Rodríguez, 2022)

- (a) The projective symmetric norm of
- (1) $\hat{\otimes}_{\pi,s,N} \ell_2$ is uniformly SSD on U_s .
 - (2) $\hat{\otimes}_{\pi,s,N} \mathcal{C}_0$ is SSD on U_s (in the complex case).
- (b) The projective norm of
- (1) $\ell_2 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_2$ is uniformly SSD on U .
 - (2) $\ell_1^N \hat{\otimes}_\pi Y$ is SSD on U whenever Y is SSD.
 - (3) $\ell_1^N \hat{\otimes}_\pi \ell_p \hat{\otimes}_\pi \ell_q$ is SSD on U whenever $2 < p, q < \infty$.

Thank you
for your attention