Um "quase esquecido" (mas natural) conceito de diferenciabilidade em espaços de Banach (que não tão clássicos assim)

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Based on a few joint works with

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Classical differentiability

Classical concepts

Gâteaux differentiability: We say that the norm of a Banach is Gâteaux differentiable at $u \in S_X$ if

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  Dini's theorem: if a monotone sequence of continuous functions converges pointwise on a compact space and if the limit function is also continuous, then the convergence is uniform.

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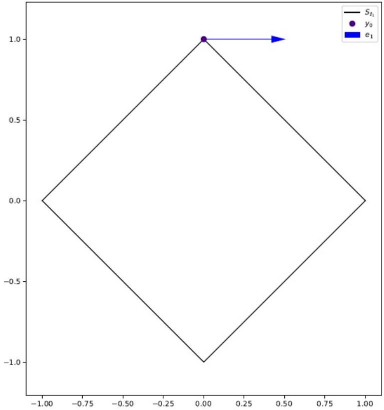

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- ⋆ If X^* is SSD, then X must be reflexive.
  Hence ℓ_1 and ℓ_∞ are *not* SSD.
  (C. Franchetti, R. Payá, 1993)

Some properties of the SSD

- ⋆ The norm of ℓ_1 is only SSD at points in S_{ℓ_1} which are sequences with finitely many nonzero terms.
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- ⋆ The norm of X is SSD when X is a predual of the
  - ⋆ Hardy space H^1 of analytic functions on the ball,
  - ⋆ Lorentz spaces $L_{p,1}(\mu)$,
  - ⋆ Trace Class $\mathcal{C}_1$.

  (S.J. Dilworth, D. Kutzarova, 1995)

Characterizations of the SSD

Theorem (C. Franchetti and R. Payá, 1993)

The following are equivalent.

- ⋆ u strongly exposes the set $D(u) = \{x^* \in X^* : \|x^*\| = x^*(u) = 1\}$.
- ⋆ D is $(n - n)$ upper semicontinuous at u.
- ⋆ For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that

$$\mathrm{dist}(D(x), D(y)) = \inf \Big\{ \|g - f\| : g \in D(x), f \in D(u) \Big\} < \varepsilon$$

  whenever $x \in S_X$ satisfies $\|x - u\| < \delta$.
- ⋆ The norm of X is SSD at u.
- ⋆ u is a τ-point of X.

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When are the norms of

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Projective tensor products

The projective tensor product $X \widehat{\otimes}_\pi Y$ is defined as the completion of the algebraic tensor product $X \otimes Y$ endowed with the norm

$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

where the infimum is taken over all representations $z = \sum_{i=1}^{n} x_i \otimes y_i$.

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- ⋆ $B_{X \widehat{\otimes}_\pi Y} = \overline{\text{co}}(B_X \otimes B_Y)$.
- ⋆ $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X \times Y, \mathbb{K})$.

Symmetric tensor products

The projective symmetric tensor product $\widehat{\otimes}_{\pi,s,N} X$ is the completion of the linear space $\otimes_{\pi,s,N} X$ generated by $\{\otimes^N z : z \in X\}$ endowed with the norm

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- $\star$ **"Dual version" property:** Given $\varepsilon > 0$ and $x^* \in S_{X^*}$, there exists $\eta(\varepsilon, x^*) > 0$ such that whenever $x \in S_X$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x^*)$, there exists $y \in S_X$ such that $|x^*(y)| = 1$ and $\|y - x\| < \varepsilon$.

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Which characterization of the SSD should we use to tackle this problem?

⋆ (the norm $\|\cdot\|$ of a Banach space X is SSD at $x \in S_X$ iff)

(C1) the pair $(X, \mathbb{K})$ has the following property: given $\varepsilon > 0$ and $x \in S_X$, there exists $\eta(\varepsilon, x) > 0$ such that whenever $x^* \in S_{X^*}$ satisfies

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⋆ (dual property)

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Strong subdifferentiability of $\ell_p \widehat{\otimes}_\pi \ell_q$

Property (C2) for linear operators and multilinear mappings:

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- ⋆ If $p^{-1} + q^{-1} \geq 1$, then the main diagonal $D = \overline{\text{span}\{e_n \otimes e_n : n \in \mathbb{N}\}}$ is one-complemented in $\ell_p \widehat{\otimes}_\pi \ell_q$ and $D \overset{(1)}{=} \ell_1$.

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- ⋆ (D., S.K. Kim, H.J. Lee, M. Mazzitelli, 2020) If $2 < p, q < \infty$, then $(\ell_q \times \ell_q, \mathbb{K})$ satisfies (C1).

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Strong subdifferentiability on $\mathcal{P}(^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

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Theorem A (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $N \in \mathbb{N}$, let X be a (reflexive) Banach space with the CAP and the sequential Kadec-Klee property and let Y be a uniformly convex Banach space. Then, the following are equivalent.

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(e) (X, Y^*) has the (C2) for N-homogeneous polynomial .

SSD on $\mathcal{P}(^N X, Y^*)$

Theorem A (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $N \in \mathbb{N}$, let X be a reflexive Banach space with the CAP and the sequential Kadec-Klee property and let Y be a uniformly convex Banach space. Then, the following are equivalent.

(a) The norm of $\mathcal{P}(^N X, Y^*)$ is SSD.

(b) $\left((\widehat{\otimes}_{\pi,s,N} X) \widehat{\otimes}_\pi Y, \mathbb{K} \right)$ has the (C2) for linear functionals.

(c) $\mathcal{P}(^N X, Y^*)$ is reflexive.

(d) $\mathcal{P}(^N X, Y^*) = \mathcal{P}_{wsc}(^N X, Y^*)$.

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SSD on $\mathcal{P}(^N X, Y^*)$

Corollary A (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p, q < \infty$ and let M_1, M_2 be Orlicz functions such that $1 < \alpha_{M_i}, \beta_{M_i} < \infty$ for $i = 1, 2$. Suppose that l_{M_2} is uniformly smooth.

(i) $\mathcal{P}(^N \ell_p)$ is SSD if and only if $N < p$.

(ii) $\mathcal{P}(^N \ell_p, \ell_q)$ is SSD if and only if $Nq < p$.

(iii) $\mathcal{P}(^N l_{M_1})$ is SSD if and only if $N < \alpha_{M_1}$.

(iv) $\mathcal{P}(^N l_{M_1}, l_{M_2})$ is SSD if and only if $N\beta_{M_2} < \alpha_{M_1}$.

(v) $\mathcal{P}(^N d(w, p))$ is SSD if and only if $N < p$.

(vi) $\mathcal{P}(^N d(w, p), l_{M_2})$ is SSD if and only if $N\beta_{M_2} < p$.

SSD on $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

Theorem B (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $N \in \mathbb{N}$ and $X_1, \ldots, X_N$ be reflexive Banach spaces with Schauder bases such that $X_1, \ldots, X_{N-1}$ have the sequential Kadec-Klee property and X_N is uniformly convex. Then, the following are equivalent.

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(a) $\mathcal{L}(X_1 \times \cdots \times X_N) = \mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*)$ is SSD.

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SSD on $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

Corollary B (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p_1, \ldots, p_N, q < \infty$ and let $M_1, \ldots, M_{N+1}$ be Orlicz functions satisfying the Δ_2-condition and such that $1 < \alpha_{M_1}, \beta_{M_1}, \ldots, \alpha_{M_{N+1}}, \beta_{M_{N+1}} < \infty$. Suppose also that $l_{M_{N+1}}$ is uniformly smooth.

(i) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N})$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.

(ii) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \ell_q)$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}$.

(iii) $\mathcal{L}(l_{M_1} \times \cdots \times l_{M_N})$ is SSD iff $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_N}} < 1$.

(iv) $\mathcal{L}(l_{M_1} \times \cdots \times l_{M_N}, l_{M_{N+1}})$ is SSD iff $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_N}} < \frac{1}{\beta_{M_{N+1}}}$.

(v) $\mathcal{L}(d(w_1, p_1) \times \cdots \times d(w_N, p_N))$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.

(vi) $\mathcal{L}(d(w_1, p_1) \times \cdots \times d(w_N, p_N), l_{M_{N+1}})$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{\beta_{M_{N+1}}}$.

Uniform strong subdifferentiability on $\widehat{\bigotimes}_{\pi,s,N} X$ and $X \widehat{\otimes}_\pi Y$

Uniformly SSD on $\widehat{\otimes}_{\pi,s,N} X$ and $X \widehat{\otimes}_\pi Y$

Definition (USSD)

The norm of a Banach space X is uniformly strongly subdifferentiable on $U \subseteq S_X$ whenever the limit

$$\lim_{t \to 0^+} \frac{\|u + tz\| - 1}{t}$$

exists uniformly for $(u, z) \in U \times B_X$.

Uniformly SSD

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(J. Becerra-Guerrero, A. Rodríguez-Palacios)

Uniformly SSD on $\widehat{\otimes}_{\pi,s,N} X$ and $X \widehat{\otimes}_\pi Y$

Equivalently,

(C3) Uniformly SSD on $U \subseteq S_X$

The norm of X is uniformly SSD on $U \subseteq S_X$ if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x_0 \in U$ satisfy

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

there exists a new functional $y^* \in S_{X^*}$ such that

$$|y^*(x)| = 1 \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

(Uniformly) SSD on $\widehat{\bigotimes}_{\pi,s,N} X$ and $X \widehat{\bigotimes}_\pi Y$

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$$U = \Big\{ x_1 \otimes \cdots \otimes x_N : \ \|x_j\| = 1 \Big\} \subseteq S_{X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N},$$

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Theorem (D., Jung, Mazzitelli, Rodríguez, 2022)

(1) The symmetric projective norm of

- **(a)** $\widehat{\otimes}_{\pi,s,N} \ell_2$ is USSD on U_s.
- **(b)** $\widehat{\otimes}_{\pi,s,N} c_0$ is SSD on U_s (in the complex case).

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(2) The (full, not symmetric) projective norm of

(a) $\ell_2\hat{\otimes}_\pi \overset{N}{\cdots} \hat{\otimes}_\pi\ell_2$ is USSD on U.

(b) $c_0\widehat{\otimes}_\pi c_0$ is SSD on U (in the complex case).

(c) $\ell_1^N\widehat{\otimes}_\pi Y$ is SSD if and only if Y is SSD.

Open Problems

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⋆ **Problem 6**: (T. Russo) Is there a space such that it is nowhere SSD?

Muito obrigado
pela sua atenção!