

On the strong subdifferentiability of the homogeneous polynomials and symmetric tensor products

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Based on a few joint works with

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 - ★ Consider ϕ_n on B_X defined by

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- ★ The norm of X is SSD iff $(\phi_n)_{n=1}^{\infty}$ converges uniformly on B_X .

Classical Banach spaces

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Dini's theorem: if a monotone sequence of continuous functions converges pointwise on a compact space and if the limit function is also continuous, then the convergence is uniform.

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Hence ℓ_1 and ℓ_∞ are *not* SSD.
(C. Franchetti, R. Payá, 1993)

Some properties of the SSD

- ★ The norm of ℓ_1 is only SSD at points in S_{ℓ_1} which are sequences with finitely many nonzero terms.
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- ★ The norm of X is SSD when X is a predual of the
 - ★ Hardy space H^1 of analytic functions on the ball,
 - ★ Lorentz spaces $L_{p,1}(\mu)$,
 - ★ Trace Class \mathcal{C}_1 .(S.J. Dilworth, D. Kutzarova, 1995)

Characterizations of the SSD

Theorem (C. Franchetti and R. Payá, 1993)

The following are equivalent.

- ★ u strongly exposes the set $D(u) = \{x^* \in X^* : \|x^*\| = x^*(u) = 1\}$.
- ★ D is $(n - n)$ upper semicontinuous at u .
- ★ For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that

$$\text{dist}(D(x), D(u)) = \inf \left\{ \|g - f\| : g \in D(x), f \in D(u) \right\} < \varepsilon$$

whenever $x \in S_X$ satisfies $\|x - u\| < \delta$.

- ★ The norm of X is SSD at u .
- ★ u is a τ -point of X .

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When are the norms of

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Projective tensor products

The **projective tensor product** $X \widehat{\otimes}_\pi Y$ is defined as the completion of the algebraic tensor product $X \otimes Y$ endowed with the norm

$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i \right\}$$

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- ★ $B_{X \widehat{\otimes}_\pi Y} = \overline{\text{co}}(B_X \otimes B_Y)$.
- ★ $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X \times Y, \mathbb{K})$.

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The **projective symmetric tensor product** $\widehat{\otimes}_{\pi,s,N} X$ is the completion of the linear space $\otimes_{\pi,s,N} X$ generated by $\{\otimes^N z : z \in X\}$ endowed with the norm

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- ★ **“Dual version” property:** Given $\varepsilon > 0$ and $x^* \in S_{X^*}$, there exists $\eta(\varepsilon, x^*) > 0$ such that whenever $x \in S_X$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x^*)$, there exists $y \in S_X$ such that $|x^*(y)| = 1$ and $\|y - x\| < \varepsilon$.

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Which characterization of the SSD should we use to tackle this problem?

★ (the norm $\|\cdot\|$ of a Banach space X is SSD at $x \in S_X$ iff)

(C1) the pair (X, \mathbb{K}) has the following property: given $\varepsilon > 0$ and $x \in S_X$, there exists $\eta(\varepsilon, x) > 0$ such that whenever $x^* \in S_{X^*}$ satisfies

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★ (dual property)

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Strong subdifferentiability of $\ell_p \widehat{\otimes}_\pi \ell_q$

Property (C2) for linear operators and multilinear mappings:

- ★ (2016, [D.](#))
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- ★ If $p^{-1} + q^{-1} \geq 1$, then the main diagonal $D = \overline{\text{span}\{e_n \otimes e_n : n \in \mathbb{N}\}}$ is one-complemented in $l_p \widehat{\otimes}_\pi l_q$ and $D \stackrel{(1)}{=} l_1$.

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- ★ (D., S.K. Kim, H.J. Lee, M. Mazzitelli, 2020) If $2 < p, q < \infty$, then $(l_q \times l_q, \mathbb{K})$ satisfies (C1).

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- ★ (D., A. Rueda Zoca, 2021) $l_p \widehat{\otimes}_\pi l_q$ is SSD *exactly* when it is reflexive.

Strong subdifferentiability on
 $\mathcal{P}({}^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

SSD on $\mathcal{P}(^N X, Y^*)$

Theorem A (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $N \in \mathbb{N}$, let X be a (reflexive) Banach space with the CAP and the sequential Kadec-Klee property and let Y be a uniformly convex Banach space. Then, the following are equivalent.

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- (b) $((\widehat{\otimes}_{\pi, s, N} X) \widehat{\otimes}_{\pi} Y, \mathbb{K})$ has the (C2) for linear functionals.

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- (e) (X, Y^*) has the (C2) for N -homogeneous polynomial .

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SSD on $\mathcal{P}(^N X, Y^*)$

Corollary A (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p, q < \infty$ and let M_1, M_2 be Orlicz functions such that $1 < \alpha_{M_i}, \beta_{M_i} < \infty$ for $i = 1, 2$. Suppose that l_{M_2} is uniformly smooth.

- (i) $\mathcal{P}(^N \ell_p)$ is SSD if and only if $N < p$.
- (ii) $\mathcal{P}(^N \ell_p, \ell_q)$ is SSD if and only if $Nq < p$.
- (iii) $\mathcal{P}(^N l_{M_1})$ is SSD if and only if $N < \alpha_{M_1}$.
- (iv) $\mathcal{P}(^N l_{M_1}, l_{M_2})$ is SSD if and only if $N\beta_{M_2} < \alpha_{M_1}$.
- (v) $\mathcal{P}(^N d(w, p))$ is SSD if and only if $N < p$.
- (vi) $\mathcal{P}(^N d(w, p), l_{M_2})$ is SSD if and only if $N\beta_{M_2} < p$.

SSD on $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

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Let $N \in \mathbb{N}$ and X_1, \dots, X_N be reflexive Banach spaces with Schauder bases such that X_1, \dots, X_{N-1} have the sequential Kadec-Klee property and X_N is uniformly convex. Then, the following are equivalent.

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Corollary B (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p_1, \dots, p_N, q < \infty$ and let M_1, \dots, M_{N+1} be Orlicz functions satisfying the Δ_2 -condition and such that $1 < \alpha_{M_1}, \beta_{M_1}, \dots, \alpha_{M_{N+1}}, \beta_{M_{N+1}} < \infty$. Suppose also that $I_{M_{N+1}}$ is uniformly smooth.

- (i) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N})$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.
- (ii) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \ell_q)$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}$.
- (iii) $\mathcal{L}(I_{M_1} \times \cdots \times I_{M_N})$ is SSD iff $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_N}} < 1$.
- (iv) $\mathcal{L}(I_{M_1} \times \cdots \times I_{M_N}, I_{M_{N+1}})$ is SSD iff $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_N}} < \frac{1}{\beta_{M_{N+1}}}$.
- (v) $\mathcal{L}(d(w_1, p_1) \times \cdots \times d(w_N, p_N))$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.
- (vi) $\mathcal{L}(d(w_1, p_1) \times \cdots \times d(w_N, p_N), I_{M_{N+1}})$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{\beta_{M_{N+1}}}$.

Uniform strong subdifferentiability on
 $\widehat{\otimes}_{\pi, s, N} X$ and $X \widehat{\otimes}_{\pi} Y$

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- ★ **Problem 4:** (R. Aron) Is there an infinite-dimensional Banach space X such that $\mathcal{P}({}^N X)$ and $\mathcal{P}({}^N X^*)$ are both reflexive?

To je všechno.
Děkuji moc!