Smoothness in normed spaces

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Based on a few joint works with

⋆ Petr Hájek (Czech Technical University in Prague, Czech Republic)

⋆ Tommaso Russo (Innsbruck University, Austria)

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- (2020) Smooth norms in dense subspaces of Banach spaces
- (2021) Smooth and polyhedral norms via FBS
- (2022) Smooth norms in dense subspaces of $\ell_p(\Gamma)$ and operator ranges

The problem

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Given a Banach space $\mathcal{X}$, is there a dense subspace $\mathcal{Y}$ of $\mathcal{X}$ which admits a C^k-smooth norm?

Does this deserve some research?

Motivation

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* Our property?

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  - ⋆ Locally uniformly rotund norms
  - ⋆ Uniformly rotund norms
  - ⋆ Strictly convex norms
  - ⋆ Uniformly smooth norms and so on...

- ⋆ Our property? **Smoothness** = **"Nice differentiable norms"**

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- ⋆ It is the **same** definition as the usual one
- ⋆ Everything works pretty much as one expects:
  - All the rules coming from Calculus hold.
  - Leibniz rules hold.
  - Implicit Function theorem (we need $\mathcal{Y}$ to be complete)

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- ⋆ C^2-**smoothness**: when the second derivative of f is continuous

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- ⋆ Everything works pretty much as one expects
- ⋆ C^k-**smoothness**: ✓
- ⋆ Tools for computing: practically the same we already know

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- It is optimal in some sense

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- Much more suitable result

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- This means that smoothness **might fail**!

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- ⋆ (Deville, 1989) If $\mathcal{X}$ has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive with exact cotype $2k$, and it contains ℓ_{2k}.
    - This means that a space with a C^∞-smooth norm either contains c_0 or ℓ_p. That is, it contains a sequence space.

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⋆ ⋆ ⋆ Let us take a look at normed spaces ⋆ ⋆ ⋆

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  Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?

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- In other words, take a nonseparable bad space when it comes to differentiability. Can we build smooth norms in dense subspaces?

(D., Hájek, Russo, 2020)

Given a Banach space $\mathcal{X}$, is there a dense subspace of $\mathcal{X}$ which admits a C^k-smooth norm?

Our results

(D., Hájek, Russo, 2020)

Let $\mathcal{X}$ be a Banach space with suppression 1-unconditional Schauder basis and let $\mathcal{Y}$ be the linear span of such a basis. Then, $\mathcal{Y}$ admits a C^{∞}-smooth norm which approximates the original one.

Our results

(D., Hájek, Russo, 2022)

Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha, \varphi_\alpha\}_{\alpha \in \Gamma}$. Consider $\mathcal{Y} := \text{span}\{e_\alpha\}_{\alpha \in \Gamma}$. Then,

(i) $\mathcal{Y}$ admits a polyhedral and LFC norm.

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  - $\operatorname{span}\{e_\alpha\}_{\alpha \in \Gamma}$ is dense in $\mathcal{X}$
- ⋆ The norm $\|\cdot\|$ is **LFC** on $\mathcal{X}$ if for each $x \in S_{\mathcal{X}}$ there is an open ngh U of x, functionals $\varphi_1, \ldots, \varphi_k \in \mathcal{X}^*$ and $G : \mathbb{R}^k \to \mathbb{R}$ such that

$$\|y\| = G(\langle \varphi_1, y \rangle, \ldots, \langle \varphi_k, y \rangle), \ \forall y \in U.$$

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    - ⋆ WCG spaces
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    - ⋆ $c_0(\Gamma)$
    - ⋆ $L_1(\mu)$ for μ finite measure
    - ⋆ $C(K)$ for some K

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    - ⋆ predual of JBW*-triples
  - $\ell_\infty(\Gamma)$
  - Many more... and
  - There exist Banach spaces without FBS

What are we interested now?

How about dense subspaces different from spans of basis?

What are we interested in now?

(D., Hájek, Russo, 2023)

Let $1 \leq p < \infty$ and Γ be any infinite set. Then

$$\mathcal{Y}_p := \Big\{ y \in \ell_p(\Gamma) \colon \|y\|_q < \infty \text{ for some } q \in (0, p) \Big\} = \bigcup_{0<q<p} \ell_q(\Gamma)$$

is a dense subspace of $\ell_p(\Gamma)$ which admits a C^∞-smooth and LFC norm that approximates the $\|\cdot\|_p$-norm.

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- $\star$ Does the space of simple functions in L_1 have a smooth norm?
- $\star$ Suppose that $\mathcal{X}$ admits a C^k-smooth norm for every $k \in \mathbb{N}$. Is it true that $\mathcal{X}$ admits a C^∞-smooth norm?

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- ⋆ Can a dense hyperplane in ℓ_1 have a smooth norm?
- ⋆ Does $\ell_1(\mathfrak{c}^+)$ have a dense subspace with a C^∞-smooth norm?
- ⋆ Does the space of simple functions in L_1 have a smooth norm?
- ⋆ Suppose that $\mathcal{X}$ admits a C^k-smooth norm for every $k \in \mathbb{N}$.
  Is it true that $\mathcal{X}$ admits a C^∞-smooth norm?
- ⋆ (Main one) Is there a Banach space $\mathcal{X}$ such that *no dense* subspace of $\mathcal{X}$ has a C^1-smooth norm?

Thank you very much
for your attention!