On the strong subdifferentiability of some (not that classical) Banach spaces

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GENERALITAT VALENCIANA

Conselleria d'Educació, Universitats i Ocupació

Based on a few joint works with

- ⋆ Sun Kwang Kim (Chungbuk National University)
- ⋆ Han Ju Lee (Dongguk University)

- ⋆ Abraham Rueda Zoca (Universidad de Granada)

- ⋆ Mingu Jung (Korea Institute For Advanced Study)
- ⋆ Martin Mazzitelli (Universidad Nacional de Cuyo)
- ⋆ Jorge Tomás Rodríguez (Universidad N.C.P. Buenos Aires)

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  - ⋆ Consider ϕ_n on B_X defined by

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  - ⋆ $(\phi_n)_{n=1}^{\infty}$ is a decreasing sequence of continuous functions pointwise converging on B_X to the continuous function $\tau(u, \cdot)$.
  - ⋆ The norm of X is SSD iff $(\phi_n)_{n=1}^{\infty}$ converges uniformly on B_X.

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  (**Dini's theorem**: if a monotone sequence of continuous functions converges pointwise on a compact space and if the limit function is also continuous, then the convergence is uniform)

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  (Dini's theorem)
- ⋆ The (sup-)norm on c_0 is SSD at every point.
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- ⋆ If X^* is SSD, then X must be reflexive.
  Hence ℓ_1 and ℓ_∞ are *not* SSD.
  (C. Franchetti, R. Payá, 1993)

Some properties of the SSD

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- ⋆ The norm of X is SSD when X is a predual of the
  - ⋆ Hardy space H^1 of analytic functions on the ball,
  - ⋆ Lorentz spaces $L_{p,1}(\mu)$,
  - ⋆ Trace Class $\mathcal{C}_1$.

  (S.J. Dilworth, D. Kutzarova, 1995)

Characterizations of the SSD

Theorem (C. Franchetti and R. Payá, 1993)

The following are equivalent.

- ⋆ u strongly exposes the set $D(u) = \{x^* \in X^* : \|x^*\| = x^*(u) = 1\}$.
- ⋆ D is $(n-n)$ upper semicontinuous at u.
- ⋆ For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that

$$\text{dist}(D(x), D(y)) = \inf \left\{ \|g - f\| : g \in D(x), f \in D(u) \right\} < \varepsilon$$

  whenever $x \in S_X$ satisfies $\|x - u\| < \delta$.
- ⋆ The norm of X is SSD at u.
- ⋆ u is a τ-point of X.

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When are the norms of

$$\mathcal{P}(^N X, Y) \qquad \mathcal{L}(X_1 \times X_2, Y) \qquad X \widehat{\otimes}_\pi Y \qquad X \widehat{\otimes}_\varepsilon Y \qquad \widehat{\otimes}_{\pi,s,N} X$$

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Projective tensor products

The projective tensor product $X \widehat{\otimes}_\pi Y$ is defined as the completion of the algebraic tensor product $X \otimes Y$ endowed with the norm

$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

where the infimum is taken over all representations $z = \sum_{i=1}^{n} x_i \otimes y_i$.

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- ⋆ $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X \times Y, \mathbb{K})$.

Symmetric tensor products

The projective symmetric tensor product $\widehat{\otimes}_{\pi,s,N} X$ is the completion of the linear space $\otimes_{\pi,s,N} X$ generated by $\{\otimes^N z : z \in X\}$ endowed with the norm

$$\|z\|_{\pi,s,N} := \inf \left\{ \sum_{i=1}^{n} |\lambda_i| : z = \sum_{i=1}^{n} \lambda_i \otimes^N x_i \right\}$$

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- ⋆ **"Dual version" property:** Given $\varepsilon > 0$ and $x^* \in S_{X^*}$, there exists $\eta(\varepsilon, x^*) > 0$ such that whenever $x \in S_X$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x^*)$, there exists $y \in S_X$ such that $|x^*(y)| = 1$ and $\|y - x\| < \varepsilon$.

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Which characterization of the SSD should we use to tackle this problem?

$\star$ (the norm $\|\cdot\|$ of a Banach space X is SSD at $x \in S_X$ iff)

(C1) the pair $(X, \mathbb{K})$ has the following property: given $\varepsilon > 0$ and $x \in S_X$, there exists $\eta(\varepsilon, x) > 0$ such that whenever $x^* \in S_{X^*}$ satisfies

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⋆ (dual property)

(C2) the pair $(X, \mathbb{K})$ has the following property: given $\varepsilon > 0$ and $x^* \in S_{X^*}$, there exists $\eta(\varepsilon, x^*) > 0$ such that whenever $x \in S_X$ satisfies $|x^*(x)| > 1 - \eta(\varepsilon, x^*)$, there exists $y \in S_X$ such that $|x^*(y)| = 1$ and $\|y - x\| < \varepsilon$.

Strong subdifferentiability of $\ell_p \widehat{\otimes}_\pi \ell_q$

Property (C2) for linear operators and multilinear mappings:

- ⋆ (2016, D.)
- ⋆ (2017, Talponen)
- ⋆ (2019, Sain)
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In fact, Property (C2) was used as a tool in

$\star$ (2022, D., Jung, Roldán, Rueda Zoca)

to get results about **norm-attaining nuclear operators**.

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Theorem (D., S.K. Kim, H.J. Lee, M. Mazzitelli, 2020)

For $p, q \geq 1$, we have:

(a) if $2 < p, q < \infty$, then $\ell_p \widehat{\otimes}_\pi \ell_q$ is SSD.

(b) if $p^{-1} + q^{-1} \geq 1$ or one of p or q is 1 or ∞, then $\ell_p \widehat{\otimes}_\pi \ell_q$ is not SSD.

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"Since SSD implies Asplund, it will not be satisfied very often."

Strong subdifferentiability on $\mathcal{P}(^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

SSD on $\mathcal{P}(^N X, Y^*)$

Theorem A (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $N \in \mathbb{N}$, let X be a (reflexive) Banach space with the CAP and the sequential Kadec-Klee property and let Y be a uniformly convex Banach space. Then, the following are equivalent.

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(b) $((\widehat{\otimes}_{\pi,s,N} X) \widehat{\otimes}_\pi Y, \mathbb{K})$ has the (C2) for linear functionals.

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(c) $\mathcal{P}(^N X, Y^*)$ is reflexive.

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(e) (X, Y^*) has the (C2) for N-homogeneous polynomial .

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SSD on $\mathcal{P}(^N X, Y^*)$

Corollary A (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p, q < \infty$ and let M_1, M_2 be Orlicz functions such that $1 < \alpha_{M_i}, \beta_{M_i} < \infty$ for $i = 1, 2$. Suppose that l_{M_2} is uniformly smooth.

(i) $\mathcal{P}(^N \ell_p)$ is SSD if and only if $N < p$.

(ii) $\mathcal{P}(^N \ell_p, \ell_q)$ is SSD if and only if $Nq < p$.

(iii) $\mathcal{P}(^N l_{M_1})$ is SSD if and only if $N < \alpha_{M_1}$.

(iv) $\mathcal{P}(^N l_{M_1}, l_{M_2})$ is SSD if and only if $N\beta_{M_2} < \alpha_{M_1}$.

(v) $\mathcal{P}(^N d(w, p))$ is SSD if and only if $N < p$.

(vi) $\mathcal{P}(^N d(w, p), l_{M_2})$ is SSD if and only if $N\beta_{M_2} < p$.

SSD on $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

Theorem B (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $N \in \mathbb{N}$ and $X_1, \ldots, X_N$ be reflexive Banach spaces with Schauder bases such that $X_1, \ldots, X_{N-1}$ have the sequential Kadec-Klee property and X_N is uniformly convex. Then, the following are equivalent.

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(a) $\mathcal{L}(X_1 \times \cdots \times X_N) = \mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*)$ is SSD.

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Let $N \in \mathbb{N}$ and $X_1, \ldots, X_N$ be reflexive Banach spaces with Schauder bases such that $X_1, \ldots, X_{N-1}$ have the sequential Kadec-Klee property and X_N is uniformly convex. Then, the following are equivalent.

(a) $\mathcal{L}(X_1 \times \cdots \times X_N) = \mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*)$ is SSD.

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SSD on $\mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K})$

Corollary B (D., Jung, Mazzitelli, Rodríguez, 2022)

Let $1 < p_1, \ldots, p_N, q < \infty$ and let $M_1, \ldots, M_{N+1}$ be Orlicz functions satisfying the Δ_2-condition and such that $1 < \alpha_{M_1}, \beta_{M_1}, \ldots, \alpha_{M_{N+1}}, \beta_{M_{N+1}} < \infty$. Suppose also that $l_{M_{N+1}}$ is uniformly smooth.

(i) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N})$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.

(ii) $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \ell_q)$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}$.

(iii) $\mathcal{L}(l_{M_1} \times \cdots \times l_{M_N})$ is SSD iff $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_N}} < 1$.

(iv) $\mathcal{L}(l_{M_1} \times \cdots \times l_{M_N}, l_{M_{N+1}})$ is SSD iff $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_N}} < \frac{1}{\beta_{M_{N+1}}}$.

(v) $\mathcal{L}(d(w_1, p_1) \times \cdots \times d(w_N, p_N))$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.

(vi) $\mathcal{L}(d(w_1, p_1) \times \cdots \times d(w_N, p_N), l_{M_{N+1}})$ is SSD iff $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{\beta_{M_{N+1}}}$.

Uniform strong subdifferentiability on $\widehat{\otimes}_{\pi,s,N}X$ and $X\widehat{\otimes}_{\pi}Y$

Uniformly SSD on $\widehat{\otimes}_{\pi,s,N} X$ and $X \widehat{\otimes}_\pi Y$

Definition (USSD)

The norm of a Banach space X is uniformly strongly subdifferentiable on $U \subseteq S_X$ whenever the limit

$$\lim_{t \to 0^+} \frac{\|u + tz\| - 1}{t} = \max \left\{ \operatorname{Re} x^*(z) : x^* \in D(u) \right\} = \tau(u, z)$$

exists uniformly for $(u, z) \in U \times B_X$.
where

$$D(u) = \left\{ x^* \in X^* : \|x^*\| = x^*(u) = 1 \right\}.$$

Uniformly SSD

⋆ Connection between numerical range theory and USSD (A. Rodríguez-Palacios)

Uniformly SSD

- ⋆ Connection between numerical range theory and USSD
  (A. Rodríguez-Palacios)

- ⋆ USSD of the norm of JB*-triples
  (J. Becerra-Guerrero, A. Rodríguez-Palacios)

Equivalently,

(C3) Uniformly SSD on $U \subseteq S_X$

The norm of X is uniformly SSD on $U \subseteq S_X$ if and only if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x_0 \in U$ satisfy

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

there exists a new functional $y^* \in S_{X^*}$ such that

$$|y^*(x)| = 1 \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

(Uniformly) SSD on $\widehat{\otimes}_{\pi,s,N}X$ and $X\widehat{\otimes}_{\pi}Y$

Sketch of the proof:

(Uniformly) SSD on $\widehat{\otimes}_{\pi,s,N}X$ and $X\widehat{\otimes}_\pi Y$

Sketch of the proof: If $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$.

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$$\begin{aligned}
\tau(u, z) = \max\{\operatorname{Re}\tilde{\varphi}(z) : \tilde{\varphi} \in D(u)\} &\leq \min\{\operatorname{Re}(\varphi + \varepsilon\psi)(z) : \psi \in B_{X^*}\} \\
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Then, for $t = \frac{\delta}{2}$, we have

$$\frac{\varepsilon}{2} > \frac{\|u + tz\| - 1}{t} - \tau(u, z) \geq \frac{\operatorname{Re}\varphi(u + tz) - 1}{t} - \operatorname{Re}\varphi(z) + \varepsilon \geq \ldots \geq \frac{\varepsilon}{2}.$$

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$$U = \left\{ x_1 \otimes \cdots \otimes x_N : \ \|x_j\| = 1 \right\} \subseteq S_{X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N},$$

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Proposition

(a) $X_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi X_N$ is USSD on U iff $(X_1, \ldots, X_N, \mathbb{K})$ has (C3).

(b) $\widehat{\otimes}_{\pi,s,N} X$ is USSD on U_s iff $(X, \mathbb{K})$ has (C3) for $\mathcal{P}(^N X, \mathbb{K})$.

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Theorem (D., Jung, Mazzitelli, Rodríguez, 2022)

(1) The symmetric projective norm of

- **(a)** $\widehat{\otimes}_{\pi,s,N}\ell_2$ is USSD on U_s.
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(2) The projective norm of

- **(a)** $\ell_2\hat{\otimes}_\pi \overset{N}{\cdots} \hat{\otimes}_\pi\ell_2$ is USSD on U.
- **(b)** $c_0\widehat{\otimes}_\pi c_0$ is SSD on U (in the complex case).
- **(c)** $\ell_1^N\widehat{\otimes}_\pi Y$ is SSD if and only if Y is SSD.

Open Problems

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(2014, M. Martín) $\mathcal{K}\text{ompact} \neq \overline{\text{NA}}^{\|\cdot\|}$

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⋆ **Problem 6**: (Problem 162, Guirao, Montesinos, Zizler) Assume that X admits both an SSD equivalent norm and a Gâteaux equivalent norm. Does it admit an equivalent Fréchet differentiable norm?

¡Muchísimas gracias!