POSSIBLE LINES OF RESEARCH (PRAGUE, FEBRUARY TO APRIL 2024)

Notation

Let (M, d) be a complete metric space where we pick 0 to be the based point of it. Consider the space $\mathrm{Lip}_0(M)$ of all Lipschitz functions f on M with real values such that $f(0) = 0$. This is a Banach space endowed with the Lipschitz number of each function. Consider the evaluations functionals $\delta(x)$ given by $\delta(x)(f) = f(x)$ for every $f \in \mathrm{Lip}_0(M)$ and every $x \in M$. The Lipschitz-free space is the closed space generated by such functionals on M and we denote it by $F(M)$.

We work with Banach spaces X over a field $\mathbb{K}$, which can be the set of real numbers, $\mathbb{R}$, or the set of complex numbers, $\mathbb{C}$. We denote by S_X and B_X the unit sphere and the unit ball of the Banach space X, respectively. We denote by X^* the dual space of X. Let Y be also a Banach space. The symbol $\mathcal{L}(X, Y)$ stands for the Banach space of all bounded linear operators $T: X \longrightarrow Y$. We say that T attains its norm, or it is norm-attaining, if there is $x_0 \in S_X$ such that

$$\|T\| = \sup_{x \in B_X} \|T(x)\| = \|T(x_0)\|.$$

In other words, T is norm-attaining when the supremum in the definition of its norm is in fact a maximum. The set of all norm-attaining operators from X into Y is denoted by $\mathrm{NA}(X, Y)$. When $Y = \mathbb{K}$, we denote it simply by $\mathrm{NA}(X)$ (notice that when $Y = \mathbb{K}$, we have bounded linear functionals $x^*: X \longrightarrow \mathbb{K}$ instead and we denote by $\mathrm{NA}(X)$ the set of all bounded linear functionals which attain their norms). We denote by $\mathrm{K}(X)$ the set of all compact operators from X into Y.

Let X, Y be *normed* linear spaces. We say that the norm $\|\cdot\|$ of X is C^k-smooth if its kth Fréchet derivative exists and is continuous at every point of $X \setminus \{0\}$. The norm is said to be C^∞-smooth if this holds for every $k \in \mathbb{N}$. We denote by $\mathcal{P}(^n X; Y)$ the normed linear space of all n-homogeneous continuous polynomials from X into Y. If $U \subset X$ is an open subset, then we say that a function $f: U \longrightarrow Y$ is analytic if, for every $a \in U$, there exist $P_n \in \mathcal{P}(^n X; Y)$ $(n \in \mathbb{N} \cup \{0\})$ and $\delta > 0$ such that, for all $x \in U(a, \delta)$,

$$f(x) = \sum_{n=0}^{\infty} P_n(x - a).$$

BACKGROUND

Lipschitz-free spaces

The Lispchitz-free space $F(M)$ has several important and relevant properties such as (a). it contains an isometric copy $\delta(M)$ of M which is linearly dense; and (b). it is the predual of the space $\mathrm{Lip}_0(M)$. Besides (a) and (b), it satisfies the following universal property: (c). every Lipschitz function from M into a Banach space X can be extended to a linear operator from $F(M)$ into X.

Nowadays, thanks to the properties described in the previous paragraph, the Lipschitz-free space is a fundamental tool to the study of the geometry of Banach spaces since it allows applications from classical linear techniques to non-linear problems.

For a complete motivation on this topic, we send the reader to several different references along these lines. For instance, the seminal paper by Kantorovich and Rubinstein about optimal transport [KR], which was followed by several mathematications as Arens, de Leeuw, Eells, Johnson, Kadets and Pestov [AE,DL,J]. A systematic study about these spaces was done by Weaver [W] in 1999 and his book is still one of the most significant references in the area.

Towards Banach spaces theory, the usage of Lipschitz-free spaces was used in this theory for the first time in 2003 by Godefroy and Kalton [GK]. There, the authors study classical Banach spaces concepts as the metric approximation property and the bounded approximation property. From these results, the development of the theory on Lipschitz-free spaces increased exponentially and nowadays is one of the most in fashion topics in Banach spaces theory. Although very easy to understand and easy to follow at a first sight, the theory of Lipschitz-free spaces is new and there are lots of work to be done still in this line. Its connection with the study of the geometry of Banach spaces is the main motivation that the candidate finds to be interested in going into this recent theory and learning its techniques.

Renorming theory

Other important topic on Banach Spaces theory that the candidate started going into not long ago (in fact, when he went to the Czech Republic for a researcher position to collaborate with Petr Hájek) is Renorming Theory, where Hájek is one of the most powerful experts nowadays around the whole world. Together with Tommaso Russo (Innsbruck University, Austria), we study the behavior of the norms of the spaces when it comes to smoothness in (not necessarily complete) normed spaces. The main question that we face here in this ongoing project is the following (very general) one.

Problem 1: Given a Banach space X and $k \in \mathbb{N} \cup \{\infty, \omega\}$, is there a dense subspace Y of X such that Y admits a C^k-smooth norm?

(by definition, C^ω-smooth means analytic). Such a line of research can be traced back at least to the papers [H, V] from the early nineties, where it was proved that *every* separable Banach space admits a dense subspace with a C^∞-smooth norm. In particular, for a separable normed space X the existence of a C^1-smooth norm does not imply that X^* is separable, a result that is possibly surprising at a first sight. Our goal is to push such a theory to the non-separable context and it turns out that the main result of one publication the candidate has with Petr Hájek and Tommaso Russo [DHR] asserts that every Banach space with a fundamental biorthogonal system (for instance, all Plichko space admit a fundamental biorthogonal system: WLD Banach spaces, hence all WCG spaces and in particular all reflexive ones; every $L_1(\mu)$-space and every $C(K)$-space, where K is a Valdivia compactum or an Abelian compact group, is a Plichko space) has a dense subspace with a C^∞-smooth norm as described in the next theorem.

Theorem 1: [DHR] Suppose that X has a fundamental biorthogonal system $\{e_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$. Also, consider the dense subspace Y of X given by $Y := \mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then Y admits

(i) a polyhedral and LFC norm.
(ii) a C^∞-smooth and LFC norm.
(iii) a C^∞-smooth and LFC bump.
(iv) locally finite, σ-uniformly discrete C^∞-smooth and LFC partitions of unity.
(v) a C^1-smooth LUR norm.

As it turned out, in most of the above results, the dense subspace Y of X is the linear span of a certain biorthogonal system in X. In particular, when X is separable, the subspace Y has countable dimension (namely, it is the linear span of a countable set). In an even more recent publication, we focused on the classical (long) sequence spaces and we show that it is possible to go beyond this limitation; in particular, we build C^∞-smooth norms on dense subspaces that are "large" in a some sense. More precisely, the following is our main result.

Theorem 2: [DHR1] Let $1 \le p < \infty$ and Γ be any infinite set. Then

$$Y_p := \{y \in \ell_p(\Gamma) : \|y\|_q < \infty \text{ for some } q \in (0,p)\} = \bigcup_{0<q<p} \ell_q(\Gamma)$$

is a dense subspace of $\ell_p(\Gamma)$ which admits a C^∞-smooth and LFC norm.

Norm-attaining theory

One of the most classical topics in the theory of Banach spaces is the study of norm-attaining functions. As a matter of fact, one of the most famous characterizations of reflexivity, due to R. James, is described in terms of linear functionals which attain their norms (see, for instance, Corollary 3.56 of [FHHMPZ]). In the same direction, E. Bishop and R. Phelps proved that the set of all norm-attaining linear functionals is dense in X^* (see [BP]). This means that whenever a functional $x^* \in X^*$ is given, one is able to find a norm-attaining functional $y^* \in X^*$ such that $\|y^* - x^*\| \approx 0$. This motivated J. Lindenstrauss to study the analogous problem for bounded linear operators in his seminal paper [LINDS2], where it was obtained for the first time an example of a Banach space such that the Bishop-Phelps theorem is no longer true for this class of functions. Consequently, this opened the gate for a crucial and vast research on the topic during the past fifty years in many different directions. Indeed, just to name a few, J. Bourgain, R.E. Huff, J. Johnson, W. Schachermayer, J.J. Uhl, J. Wolfe, and V. Zizler continued the study of the set of all linear operators which attain their norms; M. Acosta, R. Aron, F.J. Aguirre, Y.S. Choi, R. Payá tackled problems in the same line involving bilinear mappings; D. García and M. Maestre considered it for homogeneous polynomials; and more recently several problems on norm-attainment of Lipschitz maps were considered.

What it is known nowadays as the Bishop-Phelps-Bollobás theorem is a result due to B. Bollobás which is a strengthening of the Bishop-Phelps theorem (already mentioned before). It says that whenever $x \in S_X$ and $x^* \in S_{X^*}$ satisfy

$$|x^*(x)| > 1 - \frac{\varepsilon^2}{2}$$

where $0 < \varepsilon < \frac{1}{2}$, there are $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(y)| = 1,\ \|y^* - x^*\| < \varepsilon, \text{ and } \|y - x\| < \varepsilon.$$

The first expression means that y^* attains its norm at y; the second means simply the Bishop-Phelps theorem (that is, it gives the density of $\mathrm{NA}(X)$ in X^*); and the third (which is the new deal here) tells us that the point y is as close as we want to the initial point x where x^* "almost" attains its norm at.

Therefore, since the Bishop-Phelps-Bollobás theorem implies the Bishop-Phelps theorem, and since the Bishop-Phelps theorem is not valid in general for bounded linear operators (as Lindenstrauss showed), we cannot expect a general version of the Bollobás' theorem for operators. For this reason, many authors started studying the subset $\mathrm{NA}(X,Y)$ of all norm-attaining operators from X into Y to see when it is dense in $\mathcal{L}(X,Y)$ and, besides, when it is possible to get a Bollobás' type theorem for this class of functions.

OPEN PROBLEMS

Lipschitz-free spaces

As mentioned before, we are interested in considering a direct relation between Banach spaces concepts with the theory of Lipschitz-free spaces. For that reason, all the problems the candidate would like to take a look at are somehow related to Banach spaces theory, where he has some experience on. We would like to highlight that Hájek is currently one of the most active researchers in the area of Lipschitz-free spaces.

Two problems we would like to take into consideration is the behavior of extreme points of the unit ball of $F(M)$. In other words, we would like to tackle the following two problems.

Problems 1: (a). What are the extreme points of $B_{F(M)}$?
(b). What are the extreme points of $B_{F(M)^{**}}$?

About Problem 1.(a), it is conjectured that all extreme points are molecules. This is known to be true in several cases already and also studied for stronger properties as preserved extreme points, denting points and strongly exposed points (see [A, AG, APS, APP, AP1,GPPR, GPR]). About Problem 1.(b), only those belonging to $F(M)$ (in other words, the preserved extreme points of $B_{F(M)}$) have been studied (see [AG]).

Problem 2. What are the points of $F(M)$ where the norm is Fréchet differentiable?

These points must exist only when M is uniformly discrete and bounded [BLR]. For such M, all points are convex series of molecules [APS] and also points of Gâteaux differentiability are characterized [AR]; Gâteaux and Fréchet differentiability agree for finitely supported points, but not for infinitely supported ones. Related to this last problem is the following one.

Problem 3. What are the points of $F(M)$ where the norm is strongly subdifferentiable (strong subdifferentiability is a weaker concept than Fréchet differentiability; in fact, the norm is Fréchet differentiable at x if and only if the norm is Gâteaux differentiable and strongly subdifferentiable at x)?

We would like also to take a look at Schauder basis for these spaces. We have the following open question.

Problem 4: Do $F(\ell_p)$, $1 < p < \infty$, have a Schauder basis? Does it have at least finite-dimensional decomposition?

Hájek and Pernecká proved that $F(\ell_1)$ has a Schauder basis with basis constant at most 3 and that $F(\ell_n^1)$ has a monotone Schauder basis [HP]. But nothing is known for $F(\ell_p)$, not even for $F(\ell_2)$. A related question is whether $F(\ell_1)$ has a monotone basis. The following is also not known.

Problem 5: Does every $F(M)$, with $M \subseteq \mathbb{R}^n$, have a Schauder basis?

Renorming theory

One of our intentions is to continue the research on this topic as it seems to be very promising in the area. Indeed, we would like to continue pushing forward Problem 1 (which seems to be a very bold and broad question). For instance, some (more concrete) problems we want to look at closely and carefully are the following ones.

Problem 1: Is it true that the Banach space $C(K)$ has dense subspaces with analytic norm if and only if K is separable?

For this we would need to develop need techniques and new approaches since the ones in the literature seem (apparently) not to help much. Another question we would like to dig in is the following one, which is related to the size of the dense subspace which admits a smooth norm.

Problem 2: For every separable Banach space X there is a dense subspace Y of dimension continuum and with a C^∞-smooth norm?

The following problem seems to be the most irritating one since Problem 4 below seems to be the most natural Banach space for it but we still do not know how to tackle this problem. Once again, we would need to come up with new techniques and new approaches in order to solve Problems 3 and 4.

Problem 3: Determine if there exists a Banach space without any subspace that admits a smooth norm.

In fact, as we have mentioned before, we do not know even the following question.

Problem 4: In ℓ_∞, are there dense subspaces that admit no smooth norms?

Norm-attaining theory

One of the problems we would like to tackle here is the following one, which asks whether the set NA can be dense in the finite-rank operators subset. In other words (maybe simpler to understand), we have the following question.

Problem 1: Is it true that all finite-rank operators can be approximated by norm-attaining operators?

It is worth mentioning that this problem is open even when the range space is $\mathbb{R}^2$ endowed with the Euclidean norm. For this reason, Problem 1 might be one of the most ambitious one related to this topic. Nevertheless, some different approach has been done in this direction. Recently it was shown by M. Martín that when one replaces "finite-rank operators" by "compact operators" the answer is negative in general. Even more recent is a paper of the candidate together with Rubén Medina from Granada University: in this paper, we develop a new technique (consequently, a new approach) on how to get density for weighted holomorphic functions which might be helpful in order to tackle Problem 1. Even though it seems to be ambitious, we would like to give it a try.

When dealing with problems related to the Bishop-Phelps-Bollobás theorem, one immediately realizes that each problem has a different approach since once the Banach space is changed, the techniques must be changed as well. A characterization is known for those Banach spaces Y such that operators from ℓ_1 into Y satisfy a Bishop-Phelps-Bollobás type theorem for operators. Nevertheless, nothing is known about the same question for operators from X into $C(K)$, that is, we have the following question.

Problem 2: Characterize the topological Hausdorff spaces K such that the bounded linear operators from X into $C(K)$ satisfy a Bishop-Phelps-Bollobás type theorem for operators for every Banach space X.

We do not know whether operators from $C(K)$ into $C(S)$ satisfy the Bishop-Phelps-Bollobás theorem in the *complex* case (in fact, this is unknown even for the Bishop-Phelps theorem) although in the real case the answer is affirmative (let us notice that one might think at a first glance that both complex and real cases must look like each other but this is not the case since the techniques from the real case does not help in the complex case. This means that both problems have their own interest separately). Moreover, the following is not known.

Problem 3: Is it true that operators from c_0 into ℓ_1 satisfy a Bishop-Phelps-Bollobás type theorem for operators in the real case?

We have mentioned before that Problem 1 is not true for compact operators. For that reason, the version for compact operators (see [DGMM]) of the Bollobás' theorem was studied (by considering all the involved operators as compact in the Bollobás' theorem). We still do not know the following question (being the converse of it not true in general).

Problem 4: Is it true that the Bollobás theorem (the classical one) for operators implies the same theorem for compact operators?

In fact, related to Problem 4, we would like to take a look at the following problem (that we started discussing not long ago with Rubén Medina, member of the research group of Hájek).

Problem 4a: Does the Bollobás theorem for operators imply the Bollobás theorem for finite rank operators?

Since every nuclear operator is a limit of a sequence of finite-rank operators, we were motivated (see [DJRR, DGJR]) to try taking one step further in the theory of norm-attaining by studying systematically the set of all nuclear operators which attain their nuclear norms. Given X, Y Banach spaces, we say that the tensor $z \in X \widehat{\otimes}_\pi Y$ *attains its projective norm* if there is a bounded sequence $(x_n, y_n)_n \subseteq X \times Y$ with $\sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\| < \infty$ such that $z = \sum_{n=1}^{\infty} x\_n \otimes y_n$ and $\|z\|_\pi = \sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\|$, where $\|\cdot\|\_\pi$ is the projective norm of z (for a complete background on this topic, we suggest the book [RYA]).

We have found out that the study of norm-attaining tensors are extremely related to the classical concept of attaining norms [DJRR]: if every element in $X \widehat{\otimes}_\pi Y$ attains its projective norm, then the set of all norm-attaining bilinear forms on $X \times Y$ is dense in the set of all bilinear forms on $X \times Y$. This provides lots of examples of tensors which *never* attain their projective norms and opens the gate to study the denseness of the subset of norm-attaining tensors. In the classical theory of norm-attaining, Lidenstrauss proves that when X is reflexive, then the set $\mathrm{NA}(X, Y)$ is *always* dense for every Banach space Y. For this reason, it is natural to ask the counterpart for norm-attaining tensors. Nevertheless, we still do not know the answer for it.

Problem 5: Let X, Y be reflexive Banach spaces. Is it true that the subset of norm-attaining tensors is dense in $X \widehat{\otimes}_\pi Y$?

Given a Banach space X, there is also a related concept (to tensor products) called the *symmetric* tensor product $\widehat{\otimes}_{\pi,s,N} X$ where we can also define a natural norm-attaining object (see reference 8 on the List of Publications). However, we will not enter in details here. A relevant question we have in mind that we would like to tackle in a near future is the following one.

Problem 6: Is it true that the set of all norm-attaining symmetric tensors is dense in the symmetric tensor product $\widehat{\otimes}_{\pi,s,N} X$?

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