The latest developments on norm-attaining tensors

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January 2024, Pohang, South Korea
19th ILJU School of Mathematics
65th Birthday of Prof. Yun Sung Choi

- ⋆ Spanish AEI Project
  PID2019 - 106529GB - I00/AEI/10.13039/501100011033
- ⋆ Spanish AEI Project
  PID2021-122126NB-
  C33/MCIN/AEI/10.13039/501100011033 (FEDER)
- ⋆ Generalitat Valenciana Project
  CIGE/2022/97

What do we do here?

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- ⋆ Notation and historical background

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- ⋆ Motivation

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- ⋆ Notation and historical background
- ⋆ Motivation
- ⋆ Known results

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- Notation and historical background
- Motivation
- Known results
  - Projective tensor products

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  - Projective tensor products
  - Symmetric tensor products

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- Notation and historical background
- Motivation
- Known results
  - Projective tensor products
  - Symmetric tensor products
- New results and Open Questions

Notation and Historical background

SK AND HJ HAVE DONE THE JOB ALREADY.
고마워!

Our Motivations

[J. Diestel, J. Uhl, J. Johnson, J. Wolfe, ≈ 1970]

Can compact operators be approximated by norm-attaining ones?

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There are compact operators between Banach spaces which **cannot** be approximated by norm-attaining operators.

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(Perhaps the) Main problem of theory

Can **finite-rank** operators be approximated by norm-attaining ones?

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Hello world! We are here: the Nuclear Operators!

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Known results

⋆ Nuclear Operators ⋆
⋆ Projective and Symmetric Tensors ⋆

Projective tensor products

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$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

where the infimum is taken over all representation of z.

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It is known that

- $B_{X \widehat{\otimes}_\pi Y} = \overline{\text{co}}(B_X \otimes B_Y)$
- $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$

Projective tensor products

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- $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{B}(X \times Y)$
- There is a canonical operator $J : X^* \widehat{\otimes}_\pi Y \to \mathcal{L}(X, Y)$ with $\|J\| = 1$ such that

$$u = \sum_{n=1}^{\infty} x_n^* \otimes y_n \mapsto L_u,$$

where

$$L_u(x) := \sum_{n=1}^{\infty} x_n^*(x) y_n \quad (x \in X).$$

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The operators that arise in this way are called **nuclear operators**.

Nuclear operators

We denote by $\mathcal{N}(X, Y)$ the set of all nuclear operators endowed with the norm:

$$\|T\|_N := \inf\left\{\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| : T(x) = \sum_{n=1}^{\infty} x_n^*(x)y_n\right\}$$

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References for this topic

- A. Defant and K. Floret
  - ☞ *Tensor Norms and Operator Ideals*
- A. Defant, D. García, M. Maestre and P. Sevilla-Peris
  - ☞ *Dirichlet Series and Holomorphic Functions in High Dimensions*
- K. Floret
  - ☞ *Natural norms on symmetric tensor products of normed spaces*
- R.A. Ryan
  - ☞ *Introduction to tensor products of Banach spaces*

Norm-attainment Concepts

Norm-attaining concepts

- $z \in X \widehat{\otimes}_\pi Y$ **attains its projective norm** if there is a bounded sequence $(x_n, y_n) \subseteq X \times Y$ with $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$ such that $z = \sum_{n=1}^{\infty} x_n \otimes y_n$ and that $\|z\|_\pi = \sum_{n=1}^{\infty} \|x_n\| \|y_n\|$

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- $T \in \mathcal{N}(X, Y)$ **attains its nuclear norm** if there is a bounded sequence $(x_n^*, y_n) \subseteq X^* \times Y$ with $\sum_{n=1}^\infty \|x_n^*\| \|y_n\| < \infty$ such that $T = \sum_{n=1}^\infty x_n^* \otimes y_n$ and that $\|T\|_N = \sum_{n=1}^\infty \|x_n^*\| \|y_n\|$

An not that new reference for this topic

- ⋆ R. Khalil
  - ☞ *Smooth points of unit balls of operator and function spaces*

    "exact nuclear operators"

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- $\mathrm{NA}_\pi(X,Y) = \{z \in X \widehat{\otimes}_\pi Y : z \text{ attains its projective norm}\}$
- $\mathrm{NA}_{\mathcal{N}}(X,Y) = \{T \in \mathcal{N}(X,Y) : T \text{ attains its nuclear norm}\}$

Nuclear operators and Tensors Which attain their norms

A characterization (D., Jung, Roldán, Rueda Zoca, 2022)

Let X, Y be Banach spaces. Let $z \in X \widehat{\otimes}_\pi Y$ with

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$.

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(1) $z \in \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$.

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(3) $\forall G \in S_{\mathcal{L}(X, Y^*)}$, $G(z) = \|z\|_\pi$ satisfies $G(x_n)(y_n) = 1, \forall n$.

A characterization (nuclear counterpart) (D., Jung, Roldán, Rueda Zoca, 2022)

Let X, Y be Banach spaces. Let $T \in \mathcal{N}(X, Y)$ with

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$. TFAE:

(1) $T \in \mathrm{NA}_{\mathcal{N}}(X, Y)$.

(2) $\exists G \in (\ker J)^\perp$ with $\|G\| = 1$ such that $G(x_n^*)(y_n) = 1, \forall n$.

(3) $\forall G \in (\ker J)^\perp, \|G\| = 1, G(T) = \|T\|_N \Longrightarrow G(x_n^*)(y_n) = 1, \forall n$.

Relevant examples
(D., Jung, Roldán, Rueda Zoca, 2022)

Every tensor in $X\widehat{\otimes}_\pi Y$ is norm-attaining when

- X, Y are finite dimensional
  (Minkowski-Carathéodory $= \overline{\text{co}}(B_X \times B_X) = \text{co}(B_X \otimes B_Y)$)
- $X = Y = H$ complex Hilbert spaces
- $X = \ell_1$ and for any Y
  (thanks to the canonical isometry between $\ell_1\widehat{\otimes}_\pi Y$ and $\ell_1(Y)$)

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As a consequence, every nuclear operator

- $T \in \mathcal{N}(c_0, Y) = \ell_1\widehat{\otimes}_\pi Y$ *attains its nuclear norm for every* Y
  (This should be compare to the classical theory!)

It is natural to ask whether or not the equalities

$$\mathrm{NA}_{\mathcal{N}}(X,Y) = \mathcal{N}(X,Y) \quad \text{or} \quad \mathrm{NA}_{\pi}(X\widehat{\otimes}_{\pi}Y) = X\widehat{\otimes}_{\pi}Y$$

hold for every Banach spaces X and Y.

A direct connection with the classical theory (D., Jung, Roldán, Rueda Zoca, 2022)

Let X, Y be Banach spaces. Suppose that $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$.

A direct connection with the classical theory (D., Jung, Roldán, Rueda Zoca, 2022)

Let X, Y be Banach spaces. Suppose that $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$. Then, we have that

$$\overline{\mathrm{NA}(X \times Y)}^{\|\cdot\|} = \mathcal{B}(X \times Y).$$

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An immediate consequence...

Let X, Y be Banach spaces. If $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$, then

$$\overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*).$$

Examples

$\exists z \in X \widehat{\otimes}_\pi Y$ with $z \notin \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$ when
(in the following cases $\mathrm{NA}(X \times Y)$ is *not* dense in $\mathcal{B}(X \times Y, \mathbb{K})$)

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(a) $X = L_1[0,1]$ and $Y^* =$ strictly convex without the RNP (J.J. Uhl, 1976)

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(a) $X = L_1[0,1]$ and $Y^* =$ strictly convex without the RNP
(J.J. Uhl, 1976)

(b) $X =$ Gowers space and $Y = \ell_p$
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(c) $X = Y = L_1[0,1]$
✌ (Y.S. Choi, 1997) ✌

Denseness of Nuclear Operators and Tensors Which attain their norms

- ⋆ We are interested in a property which provides many norm-attaining objects

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Theorem
(D., Jung, Roldán, Rueda Zoca, 2022)

Let X be a finite dimensional and Y uniformly convex Banach space. Then, we have that

$$\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y \quad \text{and} \quad \overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_N} = \mathcal{N}(X,Y).$$

Metric π-property (P.G. Casazza's book)

We say that a Banach space X has the **metric π-property** if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subseteq S_X$ a finite collection in the sphere, then we can find a finite dimensional subspace $M \subseteq X$ such that

- M is 1-complemented
- there exists $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$ for every $i \in \{1, \ldots, n\}$.

Banach spaces with the Metric π-property (All the classical Banach spaces have it!)

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- Banach spaces with monotone Schauder basis
- $L_p(\mu)$-spaces for any $1 \leq p < \infty$ and any measure μ
- L_1-predual spaces
- $X \oplus_a Y$, whenever X, Y satisfy the metric π-property
- $X = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$ or $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_p}$, $\forall\ 1 \leq p < \infty$, X_n satisfying the metric π-property
- $X \widehat{\otimes}_\pi Y$, whenever X, Y satisfy the metric π-property
- $X \widehat{\otimes}_\varepsilon Y$, whenever X, Y satisfy the metric π-property

Metric π-property
(D., Jung, Roldán, Rueda Zoca, 2022)

Let X be a Banach space satisfying the metric π-property. If

- Y satisfies the metric π-property or
- Y is uniformly convex

then we have that

$$\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y.$$

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As a consequence...

Let X be Banach space such that X^* satisfies the metric π-property. If

- Y satisfies the metric π-property or
- Y is uniformly convex

then we have that

$$\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_N} = \mathcal{N}(X,Y).$$

Are there tensors or nuclear operators which cannot be approximated by norm-attaining tensors?

A counterexample (D., Jung, Roldán, Rueda Zoca, 2022)

Let $\mathcal{R}$ be Read's space. There exists a subspace $\mathcal{X}$ of c_0 (which fails the approximation property) and a $\mathcal{Y}$ subspace of $\mathcal{R}$ such that the set of tensors in $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}^*$ which attain their projective norms **is not dense** in $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}^*$. In other words,

$$\overline{\mathrm{NA}_\pi(\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y})}^{\|\cdot\|_\pi} \neq \mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}.$$

Symmetric tensor products

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- Norm-attaining symmetric tensors ☺
- Symmetric tensors which never attain their norms ☺
- Denseness for all classical Banach spaces ☺
- No counterexample! ☹

New Results and Open Questions

New Results
⋆ Non-norm-attaining tensors ⋆

⋆ Due to indirect arguments, we do not have an explicit way of producing tensors in $X\widehat{\otimes}_{\pi}Y$ or $\widehat{\otimes}_{\pi,s,N}X$ which do or do not attain their norms.

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⋆ Due to indirect arguments, we do not have an explicit way of producing tensors in $X\widehat{\otimes}_\pi Y$ or $\widehat{\otimes}_{\pi,s,N}X$ which do or do not attain their norms. Nevertheless, we have the following result.

(Rueda Zoca, 2023)

Let X be a Banach space with the metric π-property and whose norm locally depends upon finitely many coordinates (for instance, $X = c_0(I)$). Let Y be a Hilbert space. Then,

⋆ $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$

⋆ $\overline{X\widehat{\otimes}_\pi Y \setminus \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$

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(D., Kirme and Jung, 2024)

Let X, Y be Banach spaces.

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- ⋆ If X, Y have property quasi-α, so does $X\widehat{\otimes}_\pi Y$.
- ⋆ [✌ Y.S. Choi ✌ and H.G. Song, 2008] If X, Y have property quasi-α, then $X\widehat{\otimes}_\pi Y$ has property A.

New Results
⋆ Operators defined on $X\widehat{\otimes}_\pi Y$ ⋆

(D., Kirme and Jung, 2024)

Let X, Y, Z be Banach spaces.

- ⋆ If $\overline{\mathrm{NA}(X \times Y, Z)}^{\|\cdot\|} = \mathcal{B}(X \times Y; Z)$. Then,

$$\overline{\mathrm{NA}(X\widehat{\otimes}_\pi Y, Z)}^{\|\cdot\|} = \mathcal{L}(X\widehat{\otimes}_\pi Y, Z).$$

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- ⋆ [R. Aron, C. Finet and E. Werner, 1995] If X, Y have the RNP, then $X\widehat{\otimes}_\pi Y$ has property A.

Operators defined on $X\widehat{\otimes}_\pi Y$

(D., Kirme and Jung, 2024)

- ⋆ Property quasi-α
- ⋆ Property quasi-β
- ⋆ Property α
- ⋆ Property β
- ⋆ Lindenstrauss property A
- ⋆ Lindenstrauss property B
- ⋆ Uniformly strongly exposed points
- ⋆ Residuality

Open Questions and New Directions

Open Questions

- ⋆ Let X be **reflexive** and Y be **finite dimensional** Banach spaces. Is every tensor in $X\widehat{\otimes}_\pi Y$ norm-attaining?

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- Let X be **reflexive** and Y be **finite dimensional** Banach spaces. Is every tensor in $X\widehat{\otimes}_\pi Y$ norm-attaining?

  [Godefroy, 2015] *If $X = L_1(\mathbb{T})$, where $\mathbb{T}$ is the unit circle equipped with the Haar measure and Y is the 2-dimensional Hilbert space ℓ_2^2, we have that*

$$\mathrm{NA}_\pi(L_1(\mathbb{T}) \otimes \ell_2^2) \neq L_1(\mathbb{T}) \otimes \ell_2^2.$$

Open Questions

- ⋆ Let X be **reflexive** and Y be **finite dimensional** Banach spaces. Is every tensor in $X\widehat{\otimes}_\pi Y$ norm-attaining?

  [Godefroy, 2015] *If $X = L_1(\mathbb{T})$, where $\mathbb{T}$ is the unit circle equipped with the Haar measure and Y is the 2-dimensional Hilbert space ℓ_2^2, we have that*

$$\mathrm{NA}_\pi(L_1(\mathbb{T}) \otimes \ell_2^2) \neq L_1(\mathbb{T}) \otimes \ell_2^2.$$

[D., García-Lirola, Jung, Rueda Zoca, 2023]

Let X be a Banach space such that $B_X = \mathrm{co}(\{x_1, \ldots, x_n\})$ for some $x_1, \ldots, x_n \in S_X$ and assume that Y is a predual space. Then,

$$\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y.$$

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New Directions

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New Directions

- **(D.L. Rodríguez-Vidanes)** How about **lineability** properties for the subset of non-norm-attaining tensors?
- **(D., Kirme, Jung)** To define a **Bollobás** type theorem for tensors (in the sense of [Acosta, Aron, García, Maestre])?

Thank You
For Your Attention

Thank You
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and wait! There is more... ✌