Norm-attaining tensors. || Sheldon Gil Dantas ||

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- What do we do here?

1. ~~[illegible]~~ Motivation × Background
2. ~~[illegible]~~ Known Results

   2.1. Projective tensor products

   2.2. Symmetric tensor products

   not much!

3. Open Problems.

   × New lines of investigation.

1. Motivation × Background

Def.: We say that a linear functional $x^* \in X^*$ attains its norm (norm-attaining) if there is $x_0 \in B_X$ such that $\|x^*\| = \sup_{x \in B_X} |x^*(x)| = |x^*(x_0)|$.

Q1 How many functionals on X attain the norm?

Examples: (a) $f : \ell_1 \longrightarrow \mathbb{K}$ given by

$$f(x) := \sum_{k=1}^{+\infty} \frac{x_k}{k} \qquad (x \in \ell_1)$$

is such that $|f(x)| \leq \sum_{k=1}^{+\infty} |x_k| = \|x\|_1$, $\forall x \in \ell_1$ and

$|f(e_1)| = 1$. This means f is norm-attaining.

(b) $f : \ell_1 \longrightarrow \mathbb{K}$ given by

$$f(x) := \sum_{k=1}^{+\infty} \left(1 - \frac{1}{k}\right) x_k$$

is such that $|f(x)| \leq \sum_{k=1}^{+\infty} \left(1 - \frac{1}{k}\right) |x_k| \leq \|x\|_1$ and

$\|f\| \geq |f(e_n)| = 1 - \frac{1}{n} \longrightarrow 1$, as $n \to +\infty$. This implies that

$\|f\| = 1$. However, one can see that

$$|f(x)| \leq \sum_{k=1}^{+\infty} \left(1 - \frac{1}{k}\right) |x_k| < \sum_{k=1}^{+\infty} |x_k| = \|x\|_1 = 1,$$

$\forall x \in S_1$. This implies that f is <u>not</u> norm-attaining.

Coming back to Q.1 we have the <u>James Theorem</u>

<u>James Theorem</u>: (1954) X is reflexive $\Longleftrightarrow$ $\forall x^* \in X^*$ is norm-attaining, that is, $NA(X) = X^*$.

Q.2 How about the denseness?

Also true!

Bishop-Phelps Theorem (1961): For every Banach space X, $\overline{NA(X)}^{\|\cdot\|} = X^*$.

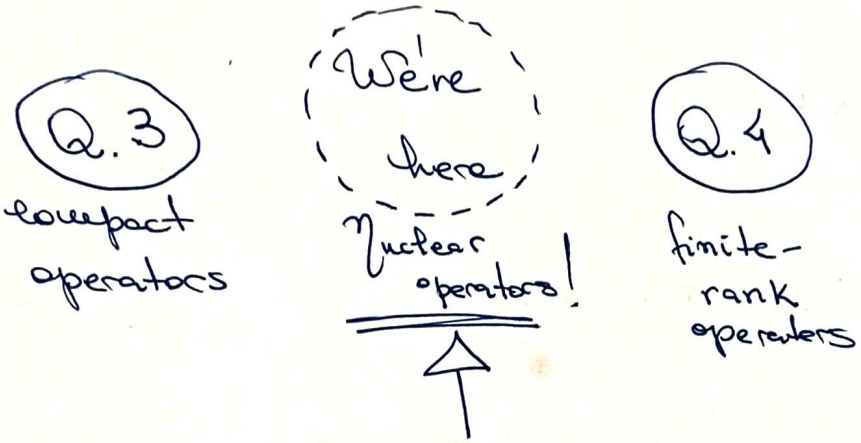
Lindenstrauss (1963): The BP theorem is no longer true for bounded linear operators!

Q.3 Can every compact operator be approximated by norm-attaing operators?

(2014, Miguel Martin) There are compact operators which cannot be approximated by norm-attaing operators!

Q.4 (Main (possibly) problem) Can every finite-rank operator be approximated by norm-attaing ones?

Since every nuclear operator is a limit of a sequence of finite-rank operators, we were motivated to give one step further in the theory:



Why the title, then? Well, because there is a clear relation between nuclear operators and tensors, which will be clear in a minute!

2. Known Results

From now on, we will be working with tensor products between Banach spaces.

* We see tensors as functionals that act on bilinear forms.
* It "linearizes" bilinear mappings in some sense.

The tensor product $X \otimes Y$ of the vector spaces X, Y can be constructed as a space of linear functionals on $B(X \times Y)$ in the following way: if $x \in X$, $y \in Y$, we denote by $x \otimes y$ the functional given by

$$(x \otimes y)(A) := \overset{\overset{\langle A, x \otimes y \rangle}{\shortparallel}}{A(x, y)}, \quad \forall A \in B(X \times Y).$$

The tensor product $X \otimes Y$ is the subspace of the dual $B(X \times Y)^{\#}$ spanned by these elements. Thus, a typical tensor in $X \otimes Y$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i, \quad n \in \mathbb{N}, \ \lambda_i \in \mathbb{K}, \ x_i \in X, y_i \in Y.$$

So, if $u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ and A is a bilinear form, then

$$u(A) = \left\langle A, \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \right\rangle = \sum_{i=1}^{n} \lambda_i A(x_i, y_i).$$

Def.: The <u>projective</u> <u>tensor</u> <u>product</u> of $\underbrace{X \text{ and } Y}_{\text{Banach spaces}}$, denoted by $X \hat{\otimes}_\pi Y$, is the completion of $X \otimes Y$ endowed with the norm

$$\|z\|_\pi = \inf\left\{ \sum_{n=1}^{+\infty} \|x_n\| \cdot \|y_n\| : \sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\| < +\infty,\ z = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}$$

Let $u \in X \hat{\otimes}_\pi Y$ and $\varepsilon > 0$. Then, there exist bounded sequences $(x_n), (y_n)$ in X, Y resp such that the series $\sum_{n=1}^{+\infty} x_n \otimes y_n$ converges to u and

$$\sum_{n=1}^{+\infty} \|x_n\| \|y_n\| < \pi(u) + \varepsilon,$$

where

$$\pi(u) = \inf\left\{ \sum_{i=1}^{N} \|x_i\| \|y_i\| : u = \sum_{i=1}^{N} x_i \otimes y_i \right\}$$

over all representations of u.

What can help us?

1. $\|x \otimes y\|_\pi = \|x\| \cdot \|y\|$, $\forall x \in X$, $\forall y \in Y$.
2. $B_{X \hat{\otimes}_\pi Y}$ is the closed convex hull of the set $B_X \otimes B_Y := \{x \otimes y : x \in B_X, y \in B_Y\}$.
3. $B(X \times Y, Z) = \mathcal{L}(X \hat{\otimes}_\pi Y, Z)$
4. $B(X \times Y) = (X \hat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$

There is a canonical operator $J : X^* \hat{\otimes}_\pi Y \to \mathcal{L}(X,Y)$

with $\|J\| = 1$ defined by

$$z = \sum_{n=1}^{+\infty} \varphi_n \otimes y_n \longmapsto L_z,$$

It might fail to be injective.

where $L_z : X \longrightarrow Y$ is given by

$$L_z(x) := \sum_{n=1}^{+\infty} \varphi_n(x) y_n \qquad (x \in X).$$

The operators that arise in this way are called <u>nuclear operators</u>. We denote the set of all nuclear operators by $\mathcal{N}(X,Y)$ endowed with the norm

$$\|T\|_N = \inf \left\{ \sum_{n=1}^{+\infty} \|x_n^*\| \|y_n\| : T(x) = \sum_{n=1}^{+\infty} x_n^*(x) y_n \right\}$$

↳ It's a Banach space!

where the inf is taken over all representations of T.

Curiosity: (i) $\|T\| \leq \|T\|_N$.
(ii) If $\dim(X) = n$, then $\|Id_X\|_N = n \geq 1 = \|Id_X\|$.

<u>Facts</u>:

1. Every nuclear operator is compact. ↳ $\forall (x_n) \subseteq X$, $(T(x_n)) \subseteq Y$ contains a converging subsequence.
2. $\mathcal{N}(X,Y) \overset{(1)}{=} X^* \hat{\otimes}_\pi Y / \operatorname{Ker} J$ isometrically
3. When X^* or Y has the AP, then $X^* \hat{\otimes}_\pi Y = \mathcal{N}(X,Y)$

Recall that X has the AP if for every compact $K \subseteq X$ and every $\varepsilon > 0$, there is a finite-rank $T : X \to X$ st $\|Tx - x\| < \varepsilon$, $\forall x \in K$.

Def.: (Norm-attaining concepts)

(a) $z \in X \hat{\otimes}_\pi Y$ attains its projective norm if there is a bounded sequence $(x_n, y_n) \subseteq X \times Y$ with $\sum_{n=1}^{+\infty} \|x_n\|\|y_n\| < +\infty$ such that $z = \sum_{n=1}^{+\infty} x_n \otimes y_n$ and that

$$\|z\|_\pi = \sum_{n=1}^{+\infty} \|x_n\|\|y_n\|.$$

In this case, z is a norm-attaining tensor.

Notation: $NA_\pi(X \hat{\otimes}_\pi Y) := \{$norm-attaining tensors$\}$.

$NA_N(X,Y) := \{$norm-attaining nuclear operators$\}$.

$NA(X,Y) := \{$norm-attaining operators$\}$.

$NA(X \times Y, Z) := \{$norm-attaining bilinear mappings$\}$.

The following results come from joint works with:

1) Minqu Jung (Seoul, Korea)

2) Óscar Roldán (Seoul, Korea)

3) Abraham Rueda Zoca. (Granada, Spain)

4) Luis Carlos García Lirola (Zaragoza, Spain)

Theorem 1: Let $z \in X \hat{\otimes}_\pi Y$ with

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$ and $y_n \in S_Y$, $\forall n \in \mathbb{N}$. TFAE:

1. $\|z\|_\pi = \sum_{n=1}^{\infty} \lambda_n$; in other words, $z \in NA_\pi(X \hat{\otimes}_\pi Y)$.

2. $\exists\ G \in \mathcal{L}(X, Y^*)$, $\|G\| = 1$: $G(x_n)(y_n) = 1$, $\forall n \in \mathbb{N}$.

3. $\forall\ G \in \mathcal{L}(X, Y^*)$, $\|G\| = 1$ with $G(z) = \|z\|_\pi$ satisfies that $G(x_n)(y_n) = 1$, $\forall n \in \mathbb{N}$.

Remark: Theorem 1 has a "nuclear" version by using $(\operatorname{Ker} J)^\perp$ but we won't treat it in this talk. We will focus on the tensors which are norm-attaing.

Example: Let X, Y be two reflexive spaces such that X^* or Y has the AP (in this case, we have $X^* \hat{\otimes}_\pi Y = \mathcal{N}(X,Y)$). Assume further that X^* is isometrically isomorphic to a subspace of Y^*. Take $G: X^* \longrightarrow Y^*$ to be a linear isometry and pick $(x_n^*) \subseteq S_{X^*}$. Now, $\forall n \in \mathbb{N}$, $\|G(x_n^*)\| = \|x_n^*\| = 1$. Since Y is reflexive, by using the James theorem, we have that $G(x_n^*) \in S_{Y^*}$ attains its norm, so there is $y_n \in S_Y$ such that $G(x_n^*)(y_n) = 1$. Now, given any sequence $(\lambda_n) \subseteq (0,1]$ with $\sum_{n=1}^{+\infty} \lambda_n < +\infty$, the nuclear operator

$$T := \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \in \mathcal{N}(X,Y)$$

attains its nuclear norm.

Q.5 A norm-attaining nuclear operator must be norm-attaining (in the classical sense)?

<u>Example</u>: Let Y be an inf-dim strictly convex Banach space. Then, there is $T \in NA_r(c_0, Y)$ such that $T \notin NA(c_0, Y)$.

Indeed, let $(y_n) \subseteq S_Y$ be linearly independent. For every $n \in \mathbb{N}$, find $y_n^* \in S_{Y^*}$ such that $y_n^*(y_n) = 1$. Define $\phi : Y \to \ell_\infty$ by $\phi(y) := (y_j^*(y))_{j=1}^{+\infty} \subseteq \ell_\infty$ $(y \in Y)$. Given $n \in \mathbb{N}$, $|y_n^*(y)| \leq \|y\|$, $\forall n \in \mathbb{N}$, which implies that

$$\sup_{n \in \mathbb{N}} |y_n^*(y)| \leq \|y\| .$$

This shows that $\phi(y) \in \ell_\infty$, $\forall y \in Y$ (i.e., ϕ is well-defined). In view of the linearity, we have that ϕ is continuous and satisfies $\|\phi\| \leq 1$. Moreover, notice that $\phi(y_n)(e_n) = 1$, $\forall n \in \mathbb{N}$, where $(e_n)_n$ is the basis of ℓ_1. This shows that $T : c_0 \longrightarrow Y$ defined by $T := \sum_{n=1}^{+\infty} \frac{1}{2^n} e_n \otimes y_n$ is a norm-attaining nuclear operator. However, since T is not finite-rank, $T \notin NA(c_0, Y)$ [Lindenstrauss 1963 or Martin 2014].

Proposition 2: Let X, Y be finite-dim spaces. Then,

$NA_\pi(X \hat{\otimes}_\pi Y) = X \hat{\otimes}_\pi Y$.

Proof: For this recall that the convex hull of a compact set is compact when X and Y are both finite-dim and so $\overline{co}(B_X \otimes B_Y) = co(B_X \otimes B_Y)$. $\blacksquare$

We've constructed before a nuclear operator $T \in \mathcal{N}(c_0, Y)$ which is norm-attaing. It turns out that any nuclear operator from c_0 into any Banach Y is norm-attaing.

Proposition 3: Let Y be a Banach space.

(a) $NA_\pi(c_0, Y) = \mathcal{N}(c_0, Y)$.

This should be compared to the classical theory: if $NA(X,Y) = \mathcal{L}(X,Y)$ for some $Y \neq \{0\}$, then X must be reflexive by the James theorem.

(b) $NA_\pi(\ell_1 \hat{\otimes}_\pi Y) = \ell_1 \hat{\otimes}_\pi Y$.

Every space with Schauder basis has the AP

In fact, we also have that:

Proposition 4: $\mathcal{N}(H,H) = NA_\pi(H,H)$ whenever H is a complex Hilbert space. Equivalently, $NA_\pi(H \hat{\otimes}_\pi H) = H \hat{\otimes}_\pi H$ as every Hilbert space has the AP.

Q.6 Is it true that $NA_\pi(X\hat{\otimes}_\pi Y) = X\hat{\otimes}_\pi Y$ for every Banach spaces X, Y? Negative!

Proposition 5 [redacted]: If $NA_\pi(X\hat{\otimes}_\pi Y) = X\hat{\otimes}_\pi Y$, then

$$\overline{NA(X\times Y)}^{\|\cdot\|} = B(X\times Y)$$

Proof: Let $\varepsilon>0$ be given. Let $B\in B(X\times Y) = (X\hat{\otimes}_\pi Y)^*$ with $\|B\|=1$. By the BP theorem, for $X\hat{\otimes}_\pi Y$, there are $B_0\in (X\hat{\otimes}_\pi Y)^*$ with $\|B_0\|=1$ and $z_0\in S_{X\hat{\otimes}_\pi Y}$ such that $\langle B_0, z_0\rangle = 1$ and $\|B_0 - B\|<\varepsilon$. By hypothesis, $z_0\in NA_\pi(X\hat{\otimes}_\pi Y)$. If we write $z_0 = \sum_{n=1}^{+\infty} \lambda_n x_n\otimes y_n$ with $\lambda_n\in\mathbb{R}^+$, $x_n\in S_X$ and $y_n\in S_Y$, we will have that $B_0(x_n,y_n)=1$, $\forall n\in\mathbb{N}$ by theorem 1. This means $B_0\in NA(X\times Y)$ and $\|B_0-B\|<\varepsilon$, that is, $\overline{NA(X\times Y)}^{\|\cdot\|} = B(X\times Y)$ as we wanted. ∎

Corollary: If $NA_\pi(X\hat{\otimes}_\pi Y) = X\hat{\otimes}_\pi Y$, then

$$\overline{NA(X, Y^*)}^{\|\cdot\|} = L(X, Y^*).$$

Thanks to that, we are now able to ~~present~~ present examples of Banach spaces X, Y such that $NA_\pi(X\hat{\otimes}_\pi Y) \neq X\hat{\otimes}_\pi Y$.

Examples: (a) $X = L_1[0,1]$

Y such that Y^* is strictly convex without the RNP

[Uhl, 1976]

(b) X = Gowers space

$Y = \ell_p$, $1 < p < +\infty$

[Gowers, 1990]

(c) $X = Y = L_1[0,1]$

[Choi, 1997]

So far, we have seen that:

- there are many norm-attaing tensors
- there is a direct connection between this new theory and the classical norm-attaing theory.
- there are non-norm-attaing tensors.

So, the natural question is:

Q.7 Can we get Bishop-Phelps for tensors? (that is, denseness of the norm-attaining tensors).

Let us talk about $\boxed{\text{denseness}}$

Let us observe the following steps:

1. Suppose that $z \in NA_\pi(X \hat{\otimes}_\pi Y)$.

2. Then, $\exists\ (x_n, y_n) \subseteq X \times Y$ bounded such that

$$\|z\|_\pi = \sum_{n=1}^{+\infty} \|x_n\| \cdot \|y_n\|$$

3. What's the best way to choose a representation for z such that $\|z - u\|_\pi < \varepsilon$ for a given $u \in X \hat{\otimes}_\pi Y$?

4. By Theorem 1, we have that $\exists\ B \in \mathcal{B}(X \times Y)$ such that $B(x_n, y_n) = 1$, $\forall n \in \mathbb{N}$.

5. Therefore, we need a property which guarantees the existence of bilinear forms that attain

their norms (in the classical sense) at a lot of points $(x_n, y_n) \in S_X \times S_Y$:

* [D., Kim, Lee, Mazzitelli, 2018, 2020] Property (P) for bilinear forms.

We won't treat it here

It is too restrictive

However, this Property (P) provides the following positive results:

Theorem 6: Let X be a finite-dimensional Banach space and Y be uniformly convex. Then,

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall x, y \in S_X$ with $\|x - y\| \geq \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\| < 1 - \delta$.

(a) $\overline{NA_\pi(X \hat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \hat{\otimes}_\pi Y$.

(b) $\overline{NA_N(X, Y)}^{\|\cdot\|_\pi} = N(X, Y)$.

It is worth mentioning that not every nuclear operator defined on a finite-dimensional space attains its nuclear norm: $\exists T : (\ell_2^2, \|\cdot\|_2) \xrightarrow{\text{nuclear}} L_1(\pi)$ which does not attain its nuclear norm [Godefroy, 2015]

Are there more examples of Banach spaces X and Y such that the denseness holds true?

Yes! We will take advantage of the finite-dim case to obtain more results. However, the projective norm <u>does</u> <u>not</u> respect subspaces.

This means that if $W \leq X$ so that $W \otimes Y$ is an algebraic subspace of $X \otimes Y$, then the norm induced on $W \otimes Y$ by $X \otimes_\pi Y$ is not in general the projective norm $\pi_{W,Y}$.

But it does respect 1-complemented subspaces. So, thinking of this direction (that is, 1-complemented subspaces + finite-dimensionality) we consider the following property.

If it is a range of a norm-one projection on X

Def.: We say that X has the <u>metric π-property</u> if given $\varepsilon > 0$ and $\{x_1, \dots, x_n\} \subseteq S_X$ a finite collection in the unit sphere, then we can find a finite-dimensional 1-complemented subspace $M \subseteq X$ such that for each $i = 1, \dots, n$, there is $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$.

Theorem 7: Let X be a BS with the metric π-property.

(a) If Y satisfies also the metric π-property or

(b) If Y is uniformly convex,

then

$$\overline{NA_\pi(X \hat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \hat{\otimes}_\pi Y.$$

Proof: We prove (a). Let $u \in S_{X \hat{\otimes}_\pi Y}$ and $\varepsilon > 0$ be given. We can find bounded sequences $(\lambda_n) \subseteq \mathbb{R}^+$, $(x_n) \subseteq S_X$ and $(y_n) \subseteq S_Y$ with $u = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$ and $\sum_{n=1}^{\infty} \lambda_n < 1 + \varepsilon$. Find $k \in \mathbb{N}$ large enough so that $\|u - z\|_{\pi_{X \otimes Y}} < \varepsilon/2$ for $z := \sum_{n=1}^{k} \lambda_n x_n \otimes y_n$. Since X and Y have the metric π-property, we can find

finite-dimensional subspaces X_0 of X and Y_0 of Y which are 1-complemented and such that, for every $n \in \{1, \dots, k\}$, there are $x_n' \in X_0$ and $y_n' \in Y_0$ such that

$$\max\{\|x_n - x_n'\|, \|y_n - y_n'\|\} < \frac{\varepsilon}{4k\lambda_n}.$$

Define $z' := \sum_{n=1}^{k} \lambda_n x_n' \otimes y_n'$ and notice that

$$\|z' - z\|_{\pi_{X\hat{\otimes}Y}} < \varepsilon/2.$$

Moreover, note that $z' \in X_0 \otimes Y_0$. We have that X_0 is 1-complemented in X and Y_0 is 1-complemented in Y. Consequently, the norm of $X \hat{\otimes}_\pi Y$ agrees on $X_0 \otimes Y_0$ with the norm of $X_0 \hat{\otimes}_\pi Y_0$. In particular, $\|z'\|_{\pi_{X_0\hat{\otimes}_\pi Y_0}} = \|z'\|_{\pi_{X\hat{\otimes}_\pi Y}}$. Finally, since X_0, Y_0 are finite-dim, $z' \in NA_\pi(X_0 \hat{\otimes}_\pi Y_0)$ and then $z' \in NA_\pi(X \hat{\otimes}_\pi Y)$. Therefore,

$$\|z' - u\|_{\pi_{X\hat{\otimes}_\pi Y}} \leq \|z' - z\| + \|z - u\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\blacksquare$

Remark: We have that the metric π-property implies the metric approximation property, which in turn is stronger than the AP. So, the previous result also yields the counterpart of it for nuclear operators.

Examples: (of Banach spaces which satisfy the metric π-property):

(a) Banach spaces with Schauder basis.

(b) $L_p(\mu)$-spaces, $\forall\, 1 \le p < +\infty$ and any measure μ.

(c) $X \oplus_a Y$, whenever X, Y satisfy the metric π-property and $\|\cdot\|_a$ is an absolute norm.

(d) h_1-predual spaces. Banach spaces X such that X^* is linearly isometric to an abstract L_1-space.

(e) $X \hat{\otimes}_\pi Y$, whenever X, Y satisfy the metric π-property.

Corollary: If $X_1, \dots, X_N$ satisfy the metric π-property and Y is any Banach space, then

$$\overline{NA_\pi(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N \hat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N \hat{\otimes}_\pi Y.$$

MIRAR CUÁNTO TIEMPO QUEDA!!!

As we've mentioned before, finite-dimensionality is <u>not</u> enough to guarantee that every tensor in $X \hat{\otimes}_\pi Y$ is norm-attaining. Nevertheless, we have the following result.

(S.D., García-Lirola, Jung, Rueda Zoca).

<u>Theorem 8</u>: Let X be a Banach space with $B_X = \mathrm{co}(\{x_1, \dots, x_n\})$ for some $x_1, \dots, x_n \in S_X$ and assume that Y is a dual space. Then,

$$NA_\pi(X \hat{\otimes}_\pi Y) = X \hat{\otimes}_\pi Y.$$

By using this result we can get the following result.

= the unit ball of every finite-dimensional subspace is the convex hull of a finite set

<u>Theorem 9</u>: Let X be a Banach space which is polyhedral satisfying the metric π-property. Assume that Y is a dual. Then

$$\overline{NA_\pi(X \hat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \hat{\otimes}_\pi Y.$$

<u>Corollary</u>: If Y is a dual space, then

$$\overline{NA_\pi(c_0 \hat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = c_0 \hat{\otimes}_\pi Y.$$

Q.9 Are there tensors which cannot be approximated by norm-attaining tensors?

So far we've seen that all of the classical Banach spaces produce positive results when talking about the denseness of the norm-attaining tensors. So, it is natural to wonder Q.9.

Let us denote by $\mathcal{R}$ the Read's space, a renorming of c_0, $\mathcal{R} = (c_0, \|\cdot\|)$, which has bidual $\mathcal{R}^{**}$ strictly convex (this space was used by Martin Rmoutil to show that NA(X) does not need to contain 2-dimensional spaces). We have the following result.

<u>Theorem 10</u>: There are subspaces X of c_0 and Y of $\mathcal{R}$ such that the set of tensors in $X \hat{\otimes}_\pi Y^*$ which attain their norms is <u>not</u> dense in $X \hat{\otimes}_\pi Y^*$.

As a consequence:

<u>Corollary</u>: There are tensors of finite rank which do not attain their norms.

<u>PROOF</u>: Let X, Y^* as in Theorem 10. Then, there is $\alpha > 0$ and $z \in X \hat{\otimes}_\pi Y$ such that $\operatorname{dist}(z, NA_\pi(X \hat{\otimes}_\pi Y^*)) \geq \alpha$. Now, take u of finite-rank such that $\|z - u\|_\pi < \alpha/2$. ∎

Then, $u \notin NA_\pi(X \hat{\otimes}_\pi Y^*)$.

3. Open Problems:

① Let X be reflexive and Y be finite-dimensional. Is every tensor in $X \hat{\otimes}_\pi Y$ norm-attaining?

② Are there Banach spaces X and Y so that $NA_r(X,Y)$ is <u>not</u> dense in $\mathcal{K}(X,Y)$?

③ Is there a Banach space X so that $NA_{\pi,s,N}(\hat{\otimes}_{\pi,s,N} X)$ is <u>not</u> dense in $\hat{\otimes}_{\pi,s,N} X$?

↳ $\hat{\otimes}_{\pi,s,N} X$ is the completion of $\otimes_{\pi,s,N} X$, the space generated by $\{z^N : z \in X\}$ under $\|z\|_{\pi,s,N} := \inf\left\{ \sum_{k=1}^{n} |\lambda_k| : z = \sum_{k=1}^{n} \lambda_k x_k^N,\ n \in \mathbb{N},\ x_k \in S_X,\ \lambda_k \in \mathbb{K} \right\}$

* $(\hat{\otimes}_{\pi,s,N} X)^* = P(^N X)$

→ Our directions:

1) Operators defined on $X \hat{\otimes}_\pi Y$

(J. Kirme and M. Jung)

2) lineability of the set of non-norm-attaining tensors

(D. h. Rodriguez-Vidanes)

3) Chevet-Saphar norm-attaining tensors.

(M. Jung).

4) Density of the subset $X \hat{\otimes}_\pi Y \setminus NA_\pi(X \hat{\otimes}_\pi Y)$

(A. Rueda-Zoca).