Norm-attaining lattice homomorphisms

Sheldon Gil Dantas

Valencia University, Spain

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GENERALITAT VALENCIANA
Conselleria d'Educació, Universitats i Ocupació

This is a join work with

- Gonzalo **Martínez Cervantes**
  Murcia University, Spain
- José David **Rodríguez Abellán**
  Murcia University, Spain
- Abraham **Rueda Zoca**
  Granada University, Spain

What do we do here?

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Lattices and homomorphisms

Vector lattices

Lattices

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Vector lattices (Riesz spaces)

A **vector lattice** is a **real** vector space X that is a lattice and satisfies for all $x, y \in X$ that

- $\star$ $x \leq y \Rightarrow x + z \leq y + z, \ \forall z \in X.$
- $\star$ $\lambda x \leq \lambda y, \ \forall \lambda \geq 0.$

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Lattices and homomorphisms

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- ⋆ $x = x^+ - x^-$ (difference of disjoint elements)
  (this decomposition is unique)

Lattices and homomorphisms

Banach lattices

We say that X is a **Banach lattice** if it is a vector lattice that is also a Banach space and it satisfies for every $x, y \in X$ that

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In particular, $\|x\| = \||x|\|$.

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- $\star$ $\mathbb{R}^X := \{f : X \longrightarrow \mathbb{R}\}$ with $f \leq g \Leftrightarrow f(x) \leq g(x)$ and

$$(f \wedge g)(x) = f(x) \wedge g(x) \quad \text{and} \quad (f \vee g)(x) = f(x) \vee g(x).$$

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The following are examples of Banach lattices.

- $C(K) = \{f : K \longrightarrow \mathbb{R} : \ f \text{ continuous}\}$ where K is a compact space with the pointwise order and the supremum norm.

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- ⋆ c_0, ℓ_p and $L_p(\mu)$ for every $1 \leq p \leq \infty$ with the pointwise order and their respective norms.

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- ⋆ c_0, ℓ_p and $L_p(\mu)$ for every $1 \leq p \leq \infty$ with the pointwise order and their respective norms.
- ⋆ Most of the classical Banach spaces have a natural lattice structure.

Lattices and homomorphisms

Positive operators

If $T : X \to Y$ is a bounded linear operator between Banach lattices, we say that T **preserves the order** if

$$T(x) \leq T(y) \text{ whenever } x \leq y.$$

Equivalently, $T(x) \geq 0$ whenever $x \geq 0$. A **positive operator** is a bounded linear operator that preserves order.

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Warning: A positive operator **might not** preserve suprema and infima.

Board time!

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for every $x, y \in X$. In particular, they preserve order, that is, lattice homomorphisms are positive.

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Reference

- ⋆ [P. Meyer-Nieberg, Banach lattices, Springer-Verlag, 1991]

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There is no James theorem for this class of functions: as we can see there are nonreflexive Banach lattices such that every lattice homomorphisms attains its norm.

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Order continuous Banach lattices

A Banach lattice X is **order continuous** if and only if every monotone order bounded sequence in X is norm-convergent.

Norm-attaining lattice homomorphisms

S.D, Martínez-Cervantes, Rodríguez Abellán, Rueda Zoca

If X is order continuous, then every lattice homomorphism in X^* attains its norm.

Board time!

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- ⋆ It is well known that every compact operator defined on a reflexive Banach space attains its norm.
- ⋆ Reflexive spaces are exactly those Banach spaces in which every functional attains its norm.
- ⋆ Does the same happen in the Banach lattice setting?

Norm-attaining lattice homomorphisms

S.D, Martínez-Cervantes, Rodríguez Abellán, Rueda Zoca

Suppose that $\mathrm{Hom}(X, \mathbb{R}) \subseteq \mathrm{NA}(X, \mathbb{R})$ and that Y is an **abstract M-space**

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Example

Let $T : c_0 \to \ell_1$ be a compact lattice homomorphism defined by

$$T\left(\sum_{n=1}^{\infty} \lambda_n e_n\right) = \sum_{n=1}^{\infty} \frac{\lambda_n}{2^n} e_n$$

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$$T\left(\sum_{n=1}^{\infty} \lambda_n e_n\right) = \sum_{n=1}^{\infty} \frac{\lambda_n}{2^n} e_n \Rightarrow \begin{cases} \mathrm{Hom}(c_0,\mathbb{R}) \subseteq \mathrm{NA}(c_0,\mathbb{R}) \\ T \in K(c_0,\ell_1) \text{ but } T \notin \mathrm{NA}(c_0,\ell_1) \end{cases}$$

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⋆ When comparing properties

Banach spaces x Banach lattices

one important class of Banach lattices (which was used several times to provide counterexamples) are the free Banach lattices FBL[E], where E is a Banach space.

Non-norm-attaining lattice homomorphisms

[2018, Antonio Avilés, José Rodríguez, Pedro Tradacete]

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Let E be a Banach space. The **free Banach lattice generated by E** is a Banach lattice, denoted by $\mathrm{FBL}[E]$, together with a bounded linear operator $\phi : E \to \mathrm{FBL}[E]$ with the property that,

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Non-norm-attaining lattice homomorphisms

[2018, Antonio Avilés, José Rodríguez, Pedro Tradacete]

Let E be a Banach space. The **free Banach lattice generated by** E is a Banach lattice, denoted by $\mathrm{FBL}[E]$, together with a bounded linear operator $\phi : E \to \mathrm{FBL}[E]$ with the property that, for every Banach lattice X and every bounded operator $T : E \to X$, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}[E] \to X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.

Draw me!

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- ⋆ It can be constructed as a certain sublattice of functions on E^*.

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- ⋆ An explicit description is provided.
- ⋆ It can be constructed as a certain sublattice of functions on E^*.
- ⋆ An explicit norm is provided.

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⋆ What happens when we apply this universal property for a linear functional x^* and $\mathbb{R}$?

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⋆ Since $\|\delta_{x^*}\| = \|x^*\|$ and δ_{x^*} "extends" x^*, we have that if x^* attains its norm, then δ_{x^*} attains its norm.

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Problem

δ_{x^*} is norm-attaining $\Leftrightarrow$ x^* is norm-attaining?

Non-norm-attaining lattice homomorphisms

Property (P)

Let E be a Banach space. A non-norm-attaining functional $x^* \in E^*$ satisfies **property (P)** if the set

$$C := \{y^* \in E^* : |x^*(x)| + |y^*(x)| \leq \|x^*\|, \ \forall\, x \in B_E\}$$

satisfies that x^* is in the w^*-closure of

$$\mathbb{R}^+ C := \{\lambda y^* : \lambda > 0, y^* \in C\}.$$

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We stay that E has property (P) if every non-norm-attaining functional of E^* has property (P).

Non-norm-attaining lattice homomorphisms

S.D., Matínez-Cervantes, Rodríguez Abellán, Rueda Zoca

Let E be a Banach space and x^* be a non-norm-attaining functional with property (P). Then, δ_{x^*} is a lattice homomorphism which does not attain its norm.

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Corollary

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- ⋆ $\ell_1(\Gamma)$ has property (P).
- ⋆ any isometric predual of $\ell_1(\Gamma)$ has property (P)
- ⋆ there is a Banach space such that E fails property (P).

Bishop-Phelps-type theorems

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The question here is...

...is it possible to get Bishop-Phelps type theorems for lattice homomorphisms?

Bishop-Phelps-type theorems

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Let X be a Banach lattice and let $x^* \in \mathrm{Hom}(X, \mathbb{R})$ be a lattice homomorphism in S_{X^*} which does not attain its norm. Then,

$$\|x^* - y^*\| \geq 1, \ \forall y^* \in \mathrm{Hom}(X, \mathbb{R}) \cap \mathrm{NA}(X, \mathbb{R}).$$

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Therefore, there is no Bishop-Phelps theorem at all!

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Lattice homomorphisms might not preserve infinite suprema!

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Problem 3

Does the existence of a non-norm-attaining on a Banach lattice X imply that X contains a copy of FBL$[E]$ for some infinite dimensional Banach space E?

Thank you very much for your attention!