On non-norm-attaining operators

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Methods in Banach spaces

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- ⋆ $\exists\, X_0$ reflexive: $T = S \circ R$, $R \in \mathcal{L}(\ell_1, X_0)$ and $S \in \mathcal{L}(X_0, Y)$.
  (Davis-Figiel-Johnson-Pełczyński factorisation theorem)

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  - ⋆ X is reflexive with basis
  - ⋆ $S(v_n)$ admits no convergent subsequences.
- ⋆ $\mathcal{K}(X, Y) \neq \mathcal{L}(X, Y)$.

- This belongs to a bigger setting as follows.

D., Jung and Martínez-Cervantes, J. Inst. Math. Jussieu, 2023

Let Y be a Banach space. TFAE.

- ⋆ Y has the Schur property.
- ⋆ $\mathcal{K}(X, Y) = \mathcal{L}(X, Y)$ for every reflexive X.
- ⋆ $\mathrm{NA}(X, Y) = \mathcal{L}(X, Y)$ for every reflexive X.
- ⋆ $\mathcal{K}(E, Y) = \mathcal{L}(E, Y)$ for every E reflexive with basis.
- ⋆ $\mathrm{NA}(E, Y) = \mathcal{L}(E, Y)$ for every E reflexive with basis.

This talk is based on join **works** with

- ⋆ Javier **Falcó**
  Valencia University, Spain
- ⋆ Daniel Luis **Rodríguez Vidanes**
  Technical University of Madrid, Spain
- ⋆ Mingu **Jung**
  KIAS, South Korea

- ⋆ Gonzalo **Martínez Cervantes**
  Murcia University, Spain

What do we do here?

Contents

⋆ Searching for linear structure...

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- ⋆ Searching for linear structure...
  - ⋆ Norm-attaining functionals
  - ⋆ Non-norm-attaining functionals
  - ⋆ Norm- and non-norm-attaining operators
  - ⋆ Operators which belong to $\mathcal{L}(X,Y) \setminus \overline{\mathrm{NA}(X,Y)}$

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both belong to $\mathrm{NA}(\ell_1)$ but $x + y \notin \mathrm{NA}(\ell_1)$.

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- P. Leonetti, T. Russo and J. Somaglia (2024+)

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- $\mathrm{NA}(\ell_1)$ is **not** a linear space but it is spaceable (as $c_0 \overset{1}{\hookrightarrow} \mathrm{NA}(\ell_1)$)

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P. Bandyopadhyay and G. Godefroy, 2006

Let X be a Banach space such that B_{X^*} is w^*-sequentially compact. If $M \subseteq \mathrm{NA}(X)$ is a closed separable subspace, then M^* is the canonical quotient of X.

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For a non-trivial σ-finite measure μ, the set $\mathrm{NA}(L_1(\mu))$ is spaceable (in fact, $c_0 \overset{1}{\hookrightarrow} \mathrm{NA}(L_1(\mu))$.

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For any infinite-dimensional Banach space X, there is an isomorphic copy $\tilde{X}$ of X such that $\mathrm{NA}(\tilde{X})$ is lineable.

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D. Pellegrino and E. Texeira, 2009

Let X, Y be Banach spaces so that Y contains an isometric copy of ℓ_q for some $1 \leq q < \infty$. Then, for $x_0 \in S_X$, the set NA_{x_0} is lineable. In particular,

- $\star$ $\mathrm{NA}(X, Y)$ is lineable.
- $\star$ $\mathcal{L}(X, Y) \setminus \mathrm{NA}(X, Y)$ is lineable if it is non-empty.

Norm- and non-norm-attaining operators

- ⋆ M. Martín, J. Merí and R. Payá, 2006 There exists a Banach space X such that $\mathrm{NA}(X, c_0) \neq \mathcal{L}(X, c_0)$.

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D., Falcó, Jung, Rodríguez-Vidanes, 2024+

If X is an infinite dimensional Banach space, then c_0 is isometrically embedded in

- ⋆ $[\mathcal{L}(X, c_0) \setminus \mathrm{NA}(X, c_0)] \cup \{0\}$.
- ⋆ $\mathrm{NA}_{x_0}(X, c_0)$ for $x_0 \in S_X$.

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J. Bourgain, Israel J. Math, 1977

If X is dentable[a], then $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$ for every Y.
(in fact, it is G_δ-dense)

[a] A Banach space X is **dentable** if every nonempty bounded subset of X has slices of arbitrarily small diameter: reflexive spaces, separable dual spaces, locally uniformly convex dual spaces, $\ell_1(\Gamma)$, $H^p(\mathbb{D})$ with $1 \leq p < \infty$.

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Non-density

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- W.T. Gowers, Israel Math. J., 1990 If $w = (1/n)_{n=1}^{\infty}$ and $1 < p < \infty$, then $\overline{\mathrm{NA}(d_*(w,1), \ell_p)} \neq \mathcal{L}(d_*(w,1), \ell_p)$.

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- M.D. Acosta, Proc. Roy. Soc. Edinburgh Sec A, 1999 Give $w \in \ell_2 \setminus \ell_1$, one can construct a Banach space $X(w)$ such that $\overline{\mathrm{NA}(X(w), Y)} \neq \mathcal{L}(X(w), Y)$ for any infinite-dimensional strictly convex space Y.

Non-density

To sum it up...

Let Γ be an infinite set.

(a) (Lindenstrauss) For Y is a strictly convex renorming of the Banach space $c_0(\Gamma)$,

$$\mathcal{L}(c_0(\Gamma), Y) \setminus \overline{\mathrm{NA}(c_0(\Gamma), Y)} \neq \emptyset.$$

(b) (Gowers) For any $1 < p < \infty$,

$$\mathcal{L}(d_*(w,1), \ell_p) \setminus \overline{\mathrm{NA}(d_*(w,1), \ell_p)} \neq \emptyset.$$

(c) (Acosta) For any infinite-dimensional strictly convex Y,

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Non-density

D., Falcó, Jung, Rodríguez-Vidanes, 2024+

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(a) For Y is a strictly convex renorming of $c_0(\Gamma)$,

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(b) For any $1 < p < \infty$,

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Non-density

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Fichtenholz-Kantorovich-Hausdorff theorem

If Γ is a set of infinite cardinality κ, then there is a **family of independent subsets** $\mathcal{Y}$ of Γ of cardinality 2^κ. In other words, for any pairwise disjoint sets $\mathcal{Y}_1, \ldots, \mathcal{Y}_n \in \mathcal{Y}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ we have

$$\mathcal{Y}_1^{\varepsilon_1} \cap \ldots \cap \mathcal{Y}_n^{\varepsilon_n} \neq \emptyset$$

where $\mathcal{Y}^0 = \mathcal{Y}$ and $\mathcal{Y}^1 = \Gamma \setminus \mathcal{Y}$ for $\mathcal{Y} \subseteq \Gamma$. In fact, such a set $\mathcal{Y}_1^{\varepsilon_1} \cap \ldots \cap \mathcal{Y}_n^{\varepsilon_n}$ is **infinite**.

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Sketch of the proof:

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Then, T_F is a well-defined bounded linear operator.

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$$0 = \left(\sum_{i=1}^{m} a_i T_{F_i} \right) (e_\alpha) = a_1 e_\alpha$$

which yields a contradiction.

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$$\left\| \left(\sum_{i=1}^{m} a_i T_{F_i} \right) (e_{\alpha_k}) \right\| = \|a_1 e_{\alpha_k}\|_Y \geq C|a_1|$$

for some $C > 0$.

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$(S \in \text{NA}(c_0, Y) \Rightarrow \exists N \in \mathbb{N}\colon S(e_i) = 0, \forall i \geq N)$

In the same direction...

Avilés, Martínez-Cervantes, Rueda Zoca and Tradacete (Rev. Mat. Iberoam., 2023)

- ⋆ *Infinite dimensional spaces in SNA(M)*

D., R. Medina, A. Quilis, Ó. Roldán (Nonlinear Analysis, 2023)

- ⋆ *On isometric embeddings into SNA(M)*

G. Choi, M. Jung, H.J. Lee and Ó. Roldán 2024+

- ⋆ *Embeddings in the sets of SNA(M)*
- ⋆ *Linear structures of NA Lipschitz and their complements*

Thank you very much for your attention!