Searching for linear structures in different contexts

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GENERALITAT VALENCIANA

Conselleria d'Educació, Universitats i Ocupació

This talk is based on join papers with

- Marc **Caballer**
  Universitat de València
- William **Côrrea**
  Universidade de São Paulo
- Javier **Falcó**
  Universitat de València
- Mingu **Jung**
  Korea Institute For Advanced Study
- Daniel Luis **Rodríguez Vidanes**
  Universidad Politécnica de Madrid

What do we do here?

Contents

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Basic concepts

Lineability

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(1940, B. Levine and D. Milman)

The set of all functions of bounded variation on $[0,1]$ **does not** contain a closed infinite dimensional subspace of $C[0,1]$ endowed with the supremum norm.

Motivation and Examples

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1872, K. Weierstrass

There exists a function that is continuous everywhere but differentiable nowhere:

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There exists a function that is continuous everywhere but differentiable nowhere:

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

where $0 < a < 1$, b is any odd integer and $ab > 1 + \frac{3\pi}{2}$.

Motivation and Examples

1931, S. Banach

The set of all continuous but nowhere differentiable functions on $\mathbb{R}$ is **residual**[a] in $C(\mathbb{R})$, when endowed with the topology of uniform convergence in compacta.

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[a] it contains some dense G_δ set
(a G_δ set is a set which is a countable intersection of open sets)

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- ⋆ Nowadays:
  - Complex Analysis
  - Set Theory
  - Dynamical Systems
  - Real Analysis
  - Functional Analysis

2 DIFFERENT CONTEXTS

- **Nets x sequences**

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- ⋆ **Non-norm-attaining** bounded linear operators

- ⋆ **Nets** × **sequences**
  (2024, D. and D.L. Rodríguez-Vidanes)
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Nets x Sequences

Sequences versus Nets

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Are these results true for **nets***?*

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This net converges to 0 (in fact, we can take $a_0 = 0$ for any neighborhood of 0) but the image of the set is

$$\{0, -1, -2, -3, \ldots\}$$

which is unbounded.

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Then, $\{A_n : n \in \mathbb{Z}\}$ is a countable net that converges in norm to the operator 0 but the net is not bounded.

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Non-norm-attainment

Norm-attaining functions

Definition

Let X be a Banach space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We say that a bounded linear function $x^* : X \to \mathbb{K}$ is **norm-attaining** if there exists $x_0 \in B_X$ such that

$$\|x^*\| = \sup_{x \in B_X} |x^*(x)| = |x^*(x_0)|.$$

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$\mathrm{NA}(X) :=$ the set of all norm-attaining functionals on X.

Norm- and non-attaining functions

Examples

We have that

(a) $x^* : \ell_1 \to \mathbb{R}$ defined by

$$x^*(x) = \sum_{n=1}^{\infty} \frac{x_n}{n} \in \mathrm{NA}(\ell_1).$$

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- **James theorem**: X is reflexive if and only if $\mathrm{NA}(X) = X^*$.

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Does every dual space contain a 2-dimensional subspace consisting of norm-attaining functionals? **No!** (2015, M. Rmoutil)

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No! (1963, J. Lindenstrauss)

Norm- and non-attaining functions

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Let X, Y be Banach spaces. Let Γ be any infinite set. Let Y be a strictly convex renorming of $c_0(\Gamma)$. Then, the subset

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This provides a great amount of operators with such a property!

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Fichtenholz-Kantorovich-Hausdorff theorem

If Γ is a set of infinite cardinality κ, then there is a **family of independent subsets** $\mathcal{Y}$ of Γ of cardinality 2^κ. In other words, for any pairwise disjoint sets $\mathcal{Y}_1, \ldots, \mathcal{Y}_n \in \mathcal{Y}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ we have

$$\mathcal{Y}_1^{\varepsilon_1} \cap \ldots \cap \mathcal{Y}_n^{\varepsilon_n} \neq \emptyset$$

where $\mathcal{Y}^0 = \mathcal{Y}$ and $\mathcal{Y}^1 = \Gamma \setminus \mathcal{Y}$ for $\mathcal{Y} \subseteq \Gamma$. In fact, such a set $\mathcal{Y}_1^{\varepsilon_1} \cap \ldots \cap \mathcal{Y}_n^{\varepsilon_n}$ is **infinite**.

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THEOREM: If Y is a strictly convex renorming of c_0, then the non-empty subset $\mathcal{L}(c_0, Y) \setminus \overline{\mathrm{NA}(c_0, Y)}$ is $2^{\aleph_0}$-spaceable.

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$$T_F(e_n) := \begin{cases} e_n, & \text{when} \quad n \in F, \\ 0, & \text{otherwise} \end{cases}$$

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Then, T_F is a well-defined bounded linear operator.

What we use to prove these results?

Claim 1: $\{T_F : F \in \mathcal{Y}\}$ is linearly independent.

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$$0 = \left(\sum_{i=1}^{m} a_i T_{F_i}\right)(e_\alpha) = a_1 e_\alpha$$

which yields a contradiction.

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Claim 2: $\overline{\text{span}}\{T_F : F \in \mathcal{Y}\} \subseteq \mathcal{L}(c_0, Y) \setminus \overline{\text{NA}(c_0, Y)}$.

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$$\left\| \left(\sum_{i=1}^{m} a_i T_{F_i} \right) (e_{\alpha_k}) \right\| = \|a_1 e_{\alpha_k}\|_Y \geq C|a_1|$$

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$(S \in \text{NA}(c_0, Y) \Rightarrow \exists N \in \mathbb{N}\colon S(e_i) = 0, \forall i \geq N)$

Thank you very much for your attention!